



ELSEVIER

Journal of Economic Behavior & Organization
Vol. 61 (2006) 276–283

JOURNAL OF
Economic Behavior
& Organization

www.elsevier.com/locate/econbase

Caps on bidding in all-pay auctions: Comments on the experiments of A. Rapoport and W. Amaldoss

Emmanuel Dechenaux^{a,1}, Dan Kovenock^{b,*},
Volodymyr Lugovskyy^{c,2}

^a Department of Economics, Kent State University, Kent, OH 44242-0001, USA

^b Department of Economics, Purdue University, 403 W. State Street, West Lafayette, IN 47907-2056, USA

^c Department of Economics, University of Memphis, 400 Fogelman College Administration Building,
Memphis, TN 38152-3120, USA

Received 4 August 2003; accepted 5 November 2004

Available online 3 October 2005

Abstract

In an article published in this journal, Rapoport and Amaldoss [Rapoport, A., Amaldoss, W., 2000. Mixed strategies and iterative elimination of strongly dominated strategies: an experimental investigation of states of knowledge. *Journal of Economic Behavior and Organization* 42, 483–521] analyze symmetric and asymmetric investment games similar to two-player all-pay auctions with bid caps. In this note, we correct an error in their characterization of the set of Nash equilibria of their symmetric investment game. We discuss the implications for the analysis of data from Rapoport and Amaldoss's experiments.

© 2005 Elsevier B.V. All rights reserved.

JEL classification: C72; C92; D44

Keywords: All-pay auction; Mixed strategies; Discrete strategy space; Bid caps; Experiments

* Corresponding author. Tel.: +1 765 494 4468; fax: +1 765 494 9658.

E-mail addresses: edechena@kent.edu (E. Dechenaux), kovenock@mgmt.purdue.edu (D. Kovenock), vlgvskyy@memphis.edu (V. Lugovskyy).

¹ Tel.: +1 330 672 1106; fax: +1 330 672 9808.

² Tel.: +1 901 678 2473; fax: +1 901 678 2685.

1. Introduction

In a recent article in this journal, [Rapoport and Amaldoss \(2000\)](#) analyze an investment game of complete information in which two firms compete for a patent or the monopoly rights to a product. The firms compete by simultaneously investing in R&D, the level of which may be chosen from a common finite grid of feasible R&D levels. Both firms face budget constraints (endowments) that are common knowledge and that place upper bounds on each firm's feasible amount invested. The firm that invests the higher amount wins the patent or monopoly right (henceforth the "prize") and earns a payoff equal to the value of the prize plus its endowment minus its investment. The firm that invests the lower amount obtains a payoff equal to its endowment minus its investment. If firms set the same level of investment, neither wins the prize and both lose their investment. It is assumed that the value of the prize is identical to the two firms, but two separate assumptions on budget constraints are examined: in Case 1 firms have identical budget constraints and in Case 2 one firm has a marginally larger budget than the other firm (US\$ 1 larger with a grid of mesh US\$ 1).

The two games examined by Rapoport and Amaldoss (henceforth R–A) are specific examples of a type of game that is known in the rent-seeking literature as an all-pay auction with bid caps. Except for the fact that R–A examine an all-pay auction with discrete strategy space and for the assumption made by R–A that in the event of a tie in expenditure both firms lose the prize, the game analyzed is very similar to the treatment of the all-pay auction with bid caps analyzed by [Che and Gale \(1998\)](#).¹

In [Rapoport and Amaldoss \(2000\)](#), the authors undertake an experimental analysis of the two cases of all-pay auctions indicated above. This note focuses on the conclusions of the authors concerning Case 1, the game in which firms have identical budget constraints. For this case, R–A claim that a unique mixed strategy Nash equilibrium exists in the game (Proposition 1) and that their experimental results lend support to several of the properties of this solution at the aggregate, although not the individual, level. In particular, R–A claim that the experimental analysis seems to indicate that the players "mixed their strategies" and that the frequency of the choice of an expenditure equal to the budget constraint increased as the prize obtained from winning increased. On the other hand, several properties of the equilibrium identified by R–A were not supported by the experimental results. One prediction of the equilibrium identified in R–A's Proposition 1 is that the expected payoff of each player is independent of the (common) value of the prize to the players. R–A's experiments with prizes of different values (but identical budget constraints) showed that the mean realized payoffs to the players increased significantly in the value of the prize. R–A also found that the incidence of subjects choosing a zero investment in the game was higher than predicted by the equilibrium, a fact that they conjectured could be a consequence of either minimax behavior or simply "mixing strategies but investing zero more often than predicted by the theory" ([Rapoport and Amaldoss, 2000](#), p. 497).

This note utilizes the results of [Dechenaux et al. \(2003a\)](#) (henceforth D–K–L) to show that the Nash equilibrium for the Investment game with symmetric budget constraints examined

¹ In [Dechenaux et al. \(2003a\)](#), we refer to the game in which both firms lose the prize in the event of a tie in expenditure as the "full dissipation" case. The case examined by Che and Gale, in which firms either split the prize or fairly randomize to determine the winner of the prize is referred to as the "partial dissipation" case.

by R–A is not generally unique. Dechenaux et al. (2003a) show that for strategy spaces comprised of an equally spaced grid of an even number of strategies strictly greater than 2 (starting with a lower bound of zero and an upper bound less than the lower of the players' valuations of the prize) there exist three mixed strategy equilibria in the two-player investment game analyzed by R–A. One of these equilibria is that identified by R–A, a symmetric equilibrium with full support and players completely dissipating all expected gains from winning the prize. The other two equilibria in the context of the symmetric game examined by Rapoport and Amaldoss (2000) involve mixed strategies with an alternating structure. One firm randomizes placing positive probability at investment levels $c_i = 0, 2, 4, \dots, e - 1$ and the other firm randomizes placing positive probability at investment levels $c_j = 1, 3, \dots, e$, where e is the common budget constraint of the two firms.²

The existence of these alternating equilibria and the possibility that some subset of experimental subjects may sometimes play the corresponding equilibrium strategies leaves open the possibility that the experimental evidence examined in Rapoport and Amaldoss (2000) is more consistent with Nash equilibrium behavior than is evident from the R–A analysis. Like the symmetric equilibrium, the alternating equilibria have the property that the frequency of the choice of an expenditure equal to the budget constraint increases as the prize obtained from winning increases. Moreover, since the alternating equilibria have the property that one player receives a positive payoff that is increasing in the common value of the prize while the other player receives zero (and therefore the utility of his endowment), aggregate behavior under these alternating equilibria will exhibit a mean payoff increasing in the value of the prize. This property is consistent with the pattern appearing in R–A's experimental data, but is not a property of the symmetric equilibrium that they identify.³ Furthermore, as long as the value of the prize r satisfies $r > e + 1$, which holds in both of the symmetric experimental treatments analyzed by R–A, the frequency of zero investments in the alternating equilibria is larger than that in the symmetric equilibrium. The frequency of zero investments in the alternating equilibria is also larger than the frequency of any specific investment level strictly between zero and the budget constraint e , a property that appears

² The error in R–A's Proposition 1 arises because proving that the system provided in (A1) and (A2) of the proof of Proposition 1 (Rapoport and Amaldoss, 2000, p. 518) has a unique solution is not sufficient to prove uniqueness of a Nash equilibrium in mixed strategies. This system assumes that the equilibrium payoff to the two players is the value of their budget, which may not hold in equilibrium. The system (A1) also is written with equality in every row. However, if the probability of choosing a particular pure strategy c is zero, the equation corresponding to that strategy in (A1) should in fact be a weak inequality (see, for instance, Baye et al. (1994) for a rigorous treatment of the programming framework for solving for the Nash equilibria of all-pay auctions with discrete strategy spaces). By imposing (A1) as it is stated, R–A implicitly eliminate all mixed strategies that do not have full support (that is, have $p(c) > 0$, for $c = 0, 1, 2, \dots, e$).

³ An associate editor of this journal questions whether it is realistic to assume that participants in a random pairwise matching experiment could coordinate on an arbitrary sequence of such asymmetric alternating Nash equilibria. Although our purpose is to show that the experimental data are compatible with some sequence of Nash equilibria, we believe that the associate editor's skepticism further bolsters the case that it is difficult to reject individual Nash behavior. For if there is difficulty in coordinating on any sequence of equilibria, then even if individual players are playing Nash strategies in any given period, the coordination failure can generate a wide range of behavior that is not supportive of joint Nash behavior either in a given period or in the aggregate. It should also be noted that in the absence of preplay communication the same skepticism is warranted for repeated play of a single equilibrium.

to be supported by the experimental data. Hence, one may not need to appeal to minimax or non-equilibrium behavior to explain the pattern of zero expenditure choices in the data.

In Section 2, we review the characterization of the set of Nash equilibria of the two-player Investment game appearing in Dechenaux et al. (2003a). Section 3 concludes by examining the implications of this characterization for the experimental results reported in Rapoport and Amaldoss (2000).

2. The model and theoretical results

Suppose there are two firms indexed by i and j . Each firm receives a similar integer endowment (or budget) $e > 1$, and each firm’s value for the prize is $r > e$. Let $C_k \equiv C = \{0, 1, 2, \dots, e\}$ denote firm k ’s (pure) strategy space, $k \in \{i, j\}$. If a firm invests a higher amount than its rival, it receives the prize and pays the amount it invested. Otherwise, it receives nothing but still pays the amount it invested. Thus, firm k ’s payoff function is

$$U_k(c_k, c_{-k}) = \begin{cases} r + e - c_k & \text{if } c_k > c_{-k}, \\ e - c_k & \text{if } c_k \leq c_{-k}. \end{cases} \quad \text{for } k \in \{i, j\}.$$

In the mixed extension of the game, let $p_k(c)$ denote a probability distribution over the elements of C . $U_k(p_k, p_{-k})$ is then firm k ’s expected payoff, $k \in \{i, j\}$.

According to R–A’s Proposition 1, the game described above has a unique mixed strategy equilibrium (see R–A, p. 488). This equilibrium is symmetric and is such that each firm earns an expected profit equal to e . Under fairly general conditions and, more importantly, under the conditions satisfied in R–A’s experiments for this game, we show below that there exist three equilibria. In the two equilibria R–A do not identify, one firm’s expected payoff is equal to r (and its rival’s is equal to e) and is therefore increasing in the reward r . These results are summarized in Claim 1:

Claim 1. *Assume $e > 1$. If e is even, there exists a unique equilibrium (p_i^*, p_j^*) characterized by*

$$p_i^*(c) = p_j^*(c) = \begin{cases} \frac{1}{r} & \text{if } c = 0, 1, \dots, e - 1 \\ 1 - \frac{e}{r} & \text{if } c = e. \end{cases} \tag{1}$$

Moreover, (p_i^*, p_j^*) satisfies $U_i^* = U_j^* = e$.

If e is odd, there exist three Nash equilibria. One equilibrium is characterized by (1) and satisfies $U_i^* = U_j^* = e$. The two other equilibria are given by (p_k^*, p_{-k}^*) , $k \in \{i, j\}$, where

$$p_k^*(c) = \begin{cases} \frac{2}{r} & \text{if } c = 1, 3 \dots, e - 2 \\ 1 - \left(\frac{e - 1}{r}\right) & \text{if } c = e \\ 0 & \text{if } c = 0, 2, \dots, e - 1 \end{cases} \tag{2}$$

and

$$p_{-k}^*(c) = \begin{cases} 1 - \left(\frac{e-1}{r}\right) & \text{if } c = 0 \\ \frac{2}{r} & \text{if } c = 2, 4, \dots, e-1 \\ 0 & \text{if } c = 1, 3, \dots, e. \end{cases} \quad (3)$$

In such equilibria, expected payoffs are given by $U_k^* = r > e$ and $U_{-k}^* = e$, $k \in \{i, j\}$.

Proof. See D–K–L (2003b). \square

The “symmetric” equilibrium of Eq. (1) and the “alternating” equilibria characterized by Eqs. (2) and (3) have several properties of empirical interest. First of all, aggregating across the two players, both types of equilibria place greater probability at an investment equal to the endowment, e , than at any one of the investment levels $1, 2, \dots, e-1$. However, the alternating equilibria place the same aggregate probability at 0 as at e , whereas the symmetric equilibrium places the same probability at 0 as at each of the levels $1, 2, \dots, e-1$. The expected payoffs aggregated across the two players also differ across the two types of equilibria. In the symmetric equilibrium, the two players dissipate all expected gains from the competition for the prize, and each earns an expected payoff equal to their common endowment level, e . This expected payoff is, of course, invariant with respect to the value of the prize, r . In the alternating equilibria one player earns an expected payoff equal to r while the other earns an expected payoff equal to e . Aggregating across the two players, the mean expected payoff of a player playing an alternating equilibrium is $(r+e)/2$, an expression that is increasing in the common value of the prize, r .

3. Implications for the analysis of experimental data

Rapoport and Amaldoss’s (2000) Proposition 1 is used as a theoretical prediction against which the data from experiments are evaluated. In particular, the authors consider two games with the following parameters:

Low reward condition (Game L) : $e = 5, r = 8$,

High reward condition (Game H) : $e = 5, r = 20$.

In R–A (2000), theoretical frequency distributions of investments for both games are calculated in accordance with Proposition 1 and presented in R–A’s Table 3 along with aggregated experimental data. Note that according to our Claim 1, the symmetric equilibrium defined by R–A’s Proposition 1 is not unique in both games. Given $e = 5$, the two “alternating” equilibria defined in Claim 1 also exist in both games.

Since the frequency distribution contains information about how often each pure strategy is played in each game, in the case of two players, for every investment $c \in C$, this distribution

can be calculated as

$$f^a = \frac{p_k^{a*}(c) + p_{-k}^{a*}(c)}{2} \quad \text{for } a \in \{L, H\} \text{ and } k \in \{i, j\}. \tag{4}$$

Furthermore, we show below that for both symmetric games that R–A analyze, the predicted frequency of a zero investment is greater in alternating equilibria than it is in the symmetric equilibrium. From (1), the predicted frequency of a zero investment in the symmetric equilibrium is $1/r$. Using (2)–(4), the predicted frequency of a zero investment in either alternating equilibrium is

$$\frac{1}{2} \left(1 - \frac{e - 1}{r} \right).$$

Comparing, we obtain

$$\frac{1}{2} \left(1 - \frac{e - 1}{r} \right) > \frac{1}{r} \Leftrightarrow r > e + 1,$$

a condition that is satisfied when e is equal to 5 and r is equal to 8 or 20.

To reflect clearly the full set of theoretical predictions, we extend Table 3 of R–A to include the frequency distributions of investments for alternating equilibria. The results are presented in Table 1.

Table 1 shows that taking alternating equilibria into consideration does not affect the theoretical predictions for the frequency of investments 1–4, but it widens the range for 0 and 5. Note that comparing the experimental data with the theoretical predictions for the symmetric equilibrium, the frequencies for investments 0 and 5 have the largest absolute deviations.

Table 1
Aggregate distribution of investment (observed and predicted for all types of equilibria)

Investment	Empirical distribution			Equilibrium predictions		
	Group 1	Group 2	Both groups	Symmetric equilibrium	Alternating equilibria	Range for both types of equilibria
Game L						
$c=0$	0.116	0.222	0.169	0.125	0.250	[0.125, 0.250]
$c=1$	0.116	0.117	0.116	0.125	0.125	0.125
$c=2$	0.096	0.079	0.088	0.125	0.125	0.125
$c=3$	0.084	0.153	0.118	0.125	0.125	0.125
$c=4$	0.094	0.086	0.090	0.125	0.125	0.125
$c=5$	0.494	0.343	0.418	0.375	0.250	[0.250, 0.375]
Game H						
$c=0$	0.135	0.147	0.141	0.050	0.400	[0.050, 0.400]
$c=1$	0.065	0.044	0.055	0.050	0.050	0.050
$c=2$	0.053	0.054	0.053	0.050	0.050	0.050
$c=3$	0.040	0.066	0.053	0.050	0.050	0.050
$c=4$	0.044	0.094	0.069	0.050	0.050	0.050
$c=5$	0.662	0.594	0.628	0.750	0.400	[0.400, 0.750]

Predicted frequencies of investment for alternating equilibria were computed using f^a in Eq. (4). The observations in bold indicate that the game was the second game for the subjects.

Table 2

Mean payoffs (observed and theoretically predicted for both types of equilibria)

Game	Observed mean payoff	Equilibrium prediction for expected payoff		
		Symmetric equilibrium	Alternating equilibria	Range for both types of equilibria
L ($r=8$)	5.536	5	6.5	[5, 6.5]
H ($r=20$)	6.845	5	12.5	[5, 12.5]

In the experiments of R–A, subjects in Group 1 played 80 periods of Game L and then 80 periods of Game H, while subjects in Group 2 participated in the two games in the reverse order. In Table 1, the observations in bold indicate that the game was the second game for the subjects. If we assume that experience reduces confusion and focus only on the pool of experienced subjects, the experimental frequencies for 0 and 5 are then completely within the theoretically predicted range.

Taking alternating equilibria into consideration also extends the range of predicted mean expected payoffs. Table 2 shows that according to the symmetric equilibrium employed by R–A the predicted mean payoff is independent of the value of the prize. However, in alternating equilibria, the mean payoff is positively related to the value of the prize, a relationship that is also observed in the data.

Finally, while this note addresses the results appearing in Rapoport and Amaldoss (2000), it also demonstrates that uniqueness of equilibrium fails to hold in Rapoport and Amaldoss's (2004) n -player all-pay auction of which the two-player Investment game in Rapoport and Amaldoss (2000) is a special case.⁴ While an analysis of the n -player case is beyond the scope of the current manuscript,⁵ it is clear that arguments parallel to those made in this paper concerning the experimental implications of a larger-than-recognized set of Nash equilibria apply to the n -player case as well.

References

- Baye, M.R., Kovenock, D., de Vries, C.J., 1994. The solution to the Tullock rent-seeking game when $R > 2$: mixed strategy equilibria and mean dissipation rates. *Public Choice* 81, 363–380.
- Baye, M.R., Kovenock, D., de Vries, C.J., 1996. The all-pay auction with complete information. *Economic Theory* 8, 291–305.
- Che, Y.-K., Gale, I.L., 1998. Caps on political lobbying. *American Economic Review* 88, 643–651.
- Dechenaux, E., Kovenock, D., Lugovsky, V., 2003a. A comment on “David vs. Goliath: an analysis of asymmetric mixed-strategy games and experimental evidence”. Krannert working paper series. Working paper no. 1162. Purdue University.

⁴ See Corollary 1 in Rapoport and Amaldoss (2004).

⁵ Baye et al. (1996) show that for the symmetric all-pay auction with continuous strategy space and no budget constraints, there is a unique equilibrium for $n=2$ and a continuum of equilibria for $n=3$. An increase in the set of equilibria may arise in moving from 2 to $n>2$ players in the discrete game analyzed by Rapoport and Amaldoss (2004).

- Dechenaux, E., Kovenock, D., Lugovskyy, V., 2003b. Caps on bidding in all-pay auctions: comments on the experiments of A. Rapoport and W. Amaldoss. Krannert working paper series. Working paper no. 1161. Purdue University.
- Rapoport, A., Amaldoss, W., 2000. Mixed strategies and iterative elimination of strongly dominated strategies: an experimental investigation of states of knowledge. *Journal of Economic Behavior and Organization* 42, 483–521.
- Rapoport, A., Amaldoss, W., 2004. Mixed-strategy play in single-stage first-price all-pay auctions with symmetric players. *Journal of Economic Behavior and Organization* 54, 585–607.