

Export Subsidies, Productivity and Welfare under Firm-Level Heterogeneity *

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Abstract

In this paper we use the monopolistic competition model with heterogeneous firms to study the effect of different policies on productivity and welfare, and provide three particular policies, which allow to reach the first best allocation in the economy. We also show that an export subsidy generates an increase in productivity, but - if policy already deals with the mark-up and consumer surplus distortions that arise in this context (for example, through a subsidy on consumption of domestic varieties) - its effect on welfare is negative due to combination of falling variety and adverse terms of trade changes.

1 Introduction

Governments all over the world encourage exports in different ways.¹ There are, of course, several theoretical reasons why doing so may be in a country's best interest. There may be rents associated with some export markets, and a subsidy may be effective in increasing a country's share of those rents (Eaton and Grossman (1986)). This is the "strategic trade policy" argument for export subsidies. Alternatively, there may be positive externalities generated by exporting, such as in the presence of external "learning by exporting", although empirically it has been hard to verify the

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¹See, for example, World Trade Organization Secretariat background paper "Export subsidies" with the details on export subsidies imposed by WTO members on a wide range of the products: <http://docsonline.wto.org/DDFDocuments/t/tn/ag/S8.doc>.

significance of such externalities (Clerides, Lach and Tybout (1998)). In this paper we explore a different idea associated with the recent models of trade with heterogeneous firms. In these models firms that export are generally more productive than domestically-oriented firms, so it is conceivable that by reallocating resources from low productivity to high productivity firms, an export subsidy may increase aggregate productivity. This may be the reasoning behind the claim that, by promoting exporting firms rather than those oriented to the domestic market, industrial policy in East-Asian countries was better (or at least less distortionary) than in Latin America.

In this paper we study the effects of several policies, namely, an export subsidy and tax, an import subsidy and tariff, and a consumption subsidy and tax, on productivity and welfare in a small economy. Our model is based on Melitz (2003) and features increasing returns, product differentiation, and productivity differences across firms. It is well known that in this kind of models trade policy affects welfare through several channels. To focus on the effect of such policy through reallocation and productivity, we construct a model, in which some of these channels are “neutralized”. There is, of course, the possible effect of trade policy through the terms of trade, which in standard models generates a positive optimal tariff. This channel is neutralized in our model by considering a “small economy” that does not affect the demand function for the varieties it produces. Thus, although the individual exporters have market power, the country as a whole does not, and hence, export prices are not distorted. Another issue arises because of the mark-up charged by monopolists in the domestic market. If consumers can buy imports at the country’s opportunity cost (or international price) but must pay mark-ups on their purchases of domestically produced varieties, this creates a distortion. In the first part of the paper, we show that a consumption subsidy on domestically produced varieties equal in size to the mark-up neutralizes this distortion. Another way to reduce this distortion is to impose a small import tariff or an export tax with their effect on welfare being equivalent to that of the consumption subsidy! Finally, there is a distortion caused by foreign importers, which do not take into account a consumer surplus generated by their entry into the domestic market. Thus, a number of foreign varieties available to domestic consumers is below the optimal level. This distortion can be neutralized by an import subsidy or by imposing a small consumer tax, or by subsidizing the export. In the presence of both externalities, we derive the exact values of the optimal policies and show that the mark-up distortion prevails so that the optimal policies are a consumption subsidy or an import tariff, or an export tax. In other words, the first best allocation in the economy can be achieved using any of these three policies. Note that this result gives more flexibility to policy-makers, whose choice of policy often depends not only on economic reasons, but also on political ones.

In the second part of the paper we look at the effect of the export subsidy closer. It is obvious that in the presence of the consumption subsidy, which neutralizes mark-up and consumer surplus distortions, an export subsidy would have a negative effect on welfare. But we show that the positive productivity effect is present: an export subsidy leads to a reallocation of resources from less productive firms oriented to the domestic market to exporters, and this increases overall produc-

tivity. What other effects could then (more than) compensate for this positive productivity effect of export subsidies? We show that welfare can be decomposed into four components: productivity, terms of trade, variety, and curvature. We then prove that the product of the last three components falls with the export subsidy, and this negative effect always dominates the productivity growth effect.

The separate behavior of the terms of trade and the variety index is also interesting. There is, of course, a standard negative effect of export subsidies on the terms of trade. However, recall that we are considering a small economy that takes as given the demand function for its products. Moreover, exports may increase through the extensive margin (i.e., more varieties exported) and not only through the intensive margin (i.e., larger exports of each variety). In fact, we show that the terms of trade as we define them do not always fall with the export subsidy. Similarly, intuition would suggest that export subsidies would decrease domestic variety, as low productivity firms exit. But since the export subsidy increases imports, this may counteract the negative effect on domestic variety and allow the variety index to increase.

The idea that the nature of beneficial policies depends on the structure of the economy and the type of competition there is not new in the literature. Flam and Helpman (1987) study the effectiveness of industrial policies in the model with homogenous firms. In particular, they develop a small economy with the differentiated good sector and increasing returns to scale and the homogenous good produced under perfect competition and constant returns to scale. The latter assumption allows the authors to achieve factor price equalization across countries, but it also creates a distortion, since there is a difference in the mark-ups across two sectors. In this setting the authors show that a small tariff is always welfare improving, while the effect of export subsidy is ambiguous. To some extent, we can say that by adding firm heterogeneity our paper fills the role of Flam and Helpman (1987) for the “post-Melitz” era. However, there is an important difference in the way we model a small economy: in Flam and Helpman (1987) firms at Home do not affect the expenditure level in the differentiated good sector abroad, but they can influence the price index there. In our paper, a small economy is “small” in all ways, since home firms have no effect on both expenditure and price index abroad. Moreover, Flam and Helpman (1987) fix the number of imported foreign varieties, while we allow it to be determined endogenously to study the welfare effect of changes in total number of varieties available at Home.

Another paper, which also deals with a mark-up distortion in the presence of firm heterogeneity, is Bilbiie, Ghironi and Melitz (2006). The authors study the dynamic, stochastic, general equilibrium closed economy, in which there is a distortion caused by heterogeneity in mark-ups across consumption of differentiated goods and leisure. They show that efficiency can be restored if the government taxes leisure (or subsidizes labor supply) at a rate equal in size to a mark-up charged by producers of differentiated goods.

The rest of the paper is organized as follows. The model is laid out and the equilibrium conditions are derived in Section 2. Section 3 shows that the first best allocation in the economy can be reached

through either a consumption subsidy, or an export tax, or an import tariff. Section 4 demonstrates the effect of the export subsidy on the economy. Section 5 concludes. The details of the proofs are given in Appendix.

2 The Model

In this section we construct the model, which incorporates both the export and consumer subsidies. The import tariff can be modeled similarly, which is done in Appendix. Consider a small country with L identical agents. Each agent supplies one unit of labor and spends his income on a continuum of domestic and imported goods indexed by v and v' , respectively. Domestic and imported goods are consumed in quantities $q(v)$ and $q_m(v')$ by each agent. Preferences are given by

$$U = \left(\int_{v \in \Omega} q(v)^\rho dv + \int_{v' \in \Omega_m} q_m(v')^\rho dv' \right)^{1/\rho}, \quad 0 < \rho < 1, \quad (1)$$

where Ω and Ω_m are the sets of available domestic and imported varieties, respectively, and $\sigma = \frac{1}{1-\rho}$ is the elasticity of substitution. We assume that there is a consumption subsidy $1-\eta \geq 0$ for domestic goods, so that consumers pay $\eta p(v)$ given price $p(v)$ charged by producers. Define the price index P by $P^{1-\sigma} = \int_{v' \in \Omega_m} p_m(v')^{1-\sigma} dv' + \int_{v \in \Omega} (\eta p(v))^{1-\sigma} dv$. Then the demand for any variety is:

$$q(v) = RP^{\sigma-1} (\eta p(v))^{-\sigma} \quad \text{and} \quad q_m(v') = RP^{\sigma-1} (p_m(v'))^{-\sigma}, \quad (2)$$

where R denotes aggregate expenditure.

There is only one factor of production, labor, used by a continuum of monopolistically competitive heterogeneous firms. Each firm pays a fixed cost wf_e to enter the market, where w denotes the wage in the economy. After paying this cost, it derives its productivity draw according to the cumulative distribution function $G(\varphi)$. To simplify the analysis, we assume that the productivity distribution is Pareto, $G(\varphi) = 1 - \left(\frac{b}{\varphi}\right)^\beta$ for $\varphi \geq b$, with $\beta > \sigma$.² In addition, there is a probability $\delta < 1$ that in each period firms can be hit by a bad shock and forced to exit.

A firm with productivity level φ has a labor requirement $f + \frac{q}{\varphi}$ to produce q units of variety v for the domestic market. Thus, it has a marginal cost $\frac{w}{\varphi}$, and given the demand function from (2), it charges a price $\frac{w}{\rho\varphi}$. Then, the quantity sold domestically, the revenues, and profits from domestic sales of a firm with productivity φ are, respectively,

$$q_d(\varphi) = RP^{\sigma-1} \left(\frac{\eta w}{\rho\varphi} \right)^{-\sigma}, \quad r_d(\varphi) = RP^{\sigma-1} \eta^{-\sigma} \left(\frac{w}{\rho\varphi} \right)^{1-\sigma}, \quad \pi_d(\varphi) = \frac{r_d(\varphi)}{\sigma} - wf. \quad (3)$$

Foreign demand for domestic variety v is given by $Ap_{\text{exp}}(v)^{-\sigma}$, where A is exogenously fixed

²Note that compared to the similar assumption of $\beta > \sigma - 1$ in Melitz (2003), we assume $\beta > \sigma$, which allows us to calculate the aggregate quantities produced for the home and foreign markets.

and $p_{\text{exp}}(v)$ is the price charged by an exporter. A firm which decides to export must pay a fixed cost wf_{exp} to access the foreign market.³ Also, we assume that it receives an ad-valorem export subsidy $s > 1^4$, calculated over export revenues, so that an exporter charging price p_{exp} gets sp_{exp} for each unit sold abroad.⁵ Thus, exporters maximize

$$\pi_{\text{exp}}(\varphi) = sA(p_{\text{exp}})^{1-\sigma} - (w/\varphi)A(p_{\text{exp}})^{-\sigma} - wf_{\text{exp}},$$

and charge price $p_{\text{exp}}(\varphi) = \frac{w}{s\rho\varphi}$. The quantity exported, the revenues, and profits from exporting are, respectively,

$$q_{\text{exp}}(\varphi) = A\left(\frac{w}{\rho s\varphi}\right)^{-\sigma}, \quad r_{\text{exp}}(\varphi) = As^\sigma\left(\frac{w}{\rho\varphi}\right)^{1-\sigma}, \quad \pi_{\text{exp}}(\varphi) = \frac{r_{\text{exp}}(\varphi)}{\sigma} - wf_{\text{exp}}, \quad (4)$$

Since all profits are increasing in φ , we can define two productivity cutoffs, x and y , for domestic producers and exporters, respectively, so that only firms with productivity above x produce for the domestic market, and only firms with productivity above y export. The conditions for these cutoffs are derived from equalizing profits from each option to zero,

$$RP^{\sigma-1}\eta^{-\sigma}\left(\frac{\rho x}{w}\right)^{\sigma-1} = \sigma wf, \quad (5)$$

$$\text{(EXP) condition} \quad As^\sigma w^{1-\sigma}(\rho y)^{\sigma-1} = \sigma wf_{\text{exp}}. \quad (6)$$

We consider only equilibria with $y > x$, i.e., there are some firms that do not export, which is consistent with the empirical evidence. Specifically, firms with $\varphi \in [b, x)$ exit without production, firms with $\varphi \in [x, y)$ produce only for the domestic market, and firms with $\varphi \in [y, \infty)$ produce for both home and foreign markets. Thus, if M_e is the mass of entrants in the economy and M is the mass of active firms in the economy, then in steady state

$$(1 - G(x))M_e = \delta M,$$

i.e., new successful entrants exactly replace exiting firms. In addition, the mass of exporters is $M_{\text{exp}} = m_{\text{exp}}M$, where $m_{\text{exp}} \equiv \frac{1-G(y)}{1-G(x)}$ is the share of exporters among the whole population of active firms in the economy.

The production structure abroad is similar to that at Home: the productivity distribution of importers is given by $G(\varphi) = 1 - \left(\frac{b}{\varphi}\right)^\beta$ so that the importer with productivity φ sets the price $p_m = \frac{\gamma}{\rho\varphi}$, where γ denotes the marginal costs of production abroad (it includes foreign wage,

³Introducing per-unit trade costs would not affect our results, so we chose to leave them out to simplify notation.

⁴Note that to model an export tax τ , it is enough to assume that $s < 1$, so that $\tau = 1 - s$. All derivations are the same for any value of s .

⁵We chose to look at this particular type of subsidy, since as was empirically estimated by Das, Roberts and Tybout (2007), such policy is far more effective at stimulating exports than policies that subsidize exporters' costs of entering foreign markets.

marginal costs, possible transport costs, etc.). Moreover, the importers have to pay fixed costs of exporting denoted by F_{exp} . Under the assumption of a small economy, both γ and F_{exp} are not affected by any changes in the home country. To simplify the analysis, let us normalize the mass of all available foreign varieties M_{Foreign} to 1. Note that only some foreign firms become importers. In particular, if we denote by z the productivity level of a marginal firm, which is indifferent between importing and not, then the mass of importers is $M_m = (1 - G(z)) M_{\text{Foreign}} = \left(\frac{b}{z}\right)^\beta$. The level of z is determined from the zero profit condition for importers:

$$r_m(z) = RP^{\sigma-1} \left(\frac{\gamma}{\rho z}\right)^{1-\sigma} = \sigma F_{\text{exp}}. \quad (7)$$

Given the structure above, we can rewrite the price index at Home as

$$P^{1-\sigma} = \theta M_m \left(\frac{\rho z}{\gamma}\right)^{\sigma-1} + \theta M \left(\frac{\rho x}{\eta w}\right)^{\sigma-1}, \quad (8)$$

where $\theta \equiv \frac{\beta}{\beta - (\sigma - 1)}$. Next, following Melitz (2003), we define $\tilde{\varphi}(x) = \int_x^\infty \varphi^{\sigma-1} \mu(\varphi) d\varphi$, where $\mu(\varphi) = \frac{g(\varphi)}{1-G(x)} = \beta \frac{x^\beta}{\varphi^{\beta+1}}$. Then the expected profit from entering is given by

$$\bar{\pi} = \pi_d(\tilde{\varphi}(x)) + m_{\text{exp}} \pi_{\text{exp}}(\tilde{\varphi}(y)) = wf(\theta - 1) + wm_{\text{exp}} f_{\text{exp}}(\theta - 1).$$

The free entry condition⁶, $\frac{\bar{\pi}}{\delta}(1 - G(x)) = wf_e$, can then be written as

$$\text{(FE) condition} \quad (\theta - 1)x^{-\beta} [f + m_{\text{exp}} f_{\text{exp}}] = \frac{\delta f_e}{b^\beta}. \quad (9)$$

Now let us derive the trade balance condition. Total export revenues are $\int_y (r_{\text{exp}}(\varphi)/s) M \mu(\varphi) d\varphi = wf_{\text{exp}} M_{\text{exp}} \sigma \theta / s$, whereas the foreign international value of imports is $\int_z r_m(\varphi) M_m \mu(\varphi) d\varphi = F_{\text{exp}} M_m \sigma \theta$. The trade balance condition can then be rewritten as

$$\text{(TB) condition} \quad M_m F_{\text{exp}} = \frac{w}{s} M_{\text{exp}} f_{\text{exp}}. \quad (10)$$

We also need to derive the formula for the mass of firms in the economy. Note that the total revenue obtained by domestic producers, $M\sigma(\bar{\pi} + w(f + m_{\text{exp}} f_{\text{exp}}))$, must be equal to wL .⁷ Thus,

$$\text{(M) condition} \quad M = \frac{L}{\sigma\theta(f + m_{\text{exp}} f_{\text{exp}})} = \frac{(\theta - 1)b^\beta L}{\sigma\theta\delta f_e} x^{-\beta}, \quad (11)$$

⁶It equalizes costs of entry to the present value of average profits times the probability of successful entry.

⁷To prove, note that the total expenditure at home is $R_m + R_d = V + R_d = wL + T = wL - (s - 1)V - \frac{1-\eta}{\eta} R_d$, where R_d and R_m are expenditures on the domestic and foreign goods, respectively, and V is the value of exports. This implies $sV + R_d/\eta = wL$. Note that while consumers pay $\eta p(\varphi)$, a domestic producer with productivity φ receives only $p(\varphi)$. However, while foreign consumers pay $p_{\text{exp}}(\varphi)$, exporters receive $sp_{\text{exp}}(\varphi)$. Thus, the total revenues of domestic firms are $sV + \frac{R_d}{\eta}$, which equals wL from above.

where the last equality follows from the (FE) condition.

Finally, we want to simplify the zero profit cutoff condition for importers. Note that the total expenditures are $R = wL + T$, where T is lump sum transfers defined as

$$T = -(s-1) \int_y \frac{r_{\text{exp}}(\varphi)}{s} M \mu(\varphi) d\varphi - (1-\eta) \int_x r_d(\varphi) M \mu(\varphi) d\varphi.$$

Normalizing L to 1 and using (5) and (6), and some simplification, we obtain

$$R = w - w\sigma\theta M \left[\frac{s-1}{s} f_{\text{exp}} m_{\text{exp}} + (1-\eta) f \right]. \quad (12)$$

Using this expression and (8) in (7), we get⁸:

$$(z) \text{ condition} \quad 1 = \sigma\theta M \left[f_{\text{exp}} m_{\text{exp}} + (1-\eta) f + \frac{F_{\text{exp}}}{w} \left(\frac{\gamma}{\eta} \frac{x}{wz} \right)^{\sigma-1} \right]. \quad (13)$$

Now we have our equilibrium system of equations (6), (9), (10), (11), and (13) with five unknown variables, x, y, w, M , and z . We are interested in exploring how different policies affect welfare, which is captured by the utility of the representative consumer. To obtain a useful expression for this utility, we first introduce some definitions. Let Q_d and Q_m be the total quantity consumed of domestic and imported goods, respectively,

$$Q_d \equiv M \int_x q(\varphi) \mu(\varphi) d\varphi = f(\sigma-1) \frac{\beta}{\beta-\sigma} Mx, \quad (14)$$

$$Q_m \equiv M_m \int_z q_m(\varphi) \mu(\varphi) d\varphi = \frac{F_{\text{exp}}}{\gamma} (\sigma-1) \frac{\beta}{\beta-\sigma} M_m z, \quad (15)$$

and let $Q_{\text{consumed}} \equiv Q_m + Q_d$ be the total quantity consumed of the imported and domestic goods. Similarly, let Q_{exp} be the total quantity of goods exported,

$$Q_{\text{exp}} = M \int_y q_{\text{exp}}(\varphi) \mu(\varphi) d\varphi = f_{\text{exp}} (\sigma-1) \frac{\beta}{\beta-\sigma} M_{\text{exp}} y, \quad (16)$$

and let $Q_{\text{produced}} \equiv Q_{\text{exp}} + Q_d$ be the total quantity produced for both the domestic and foreign markets. Then the utility per capita can be expressed as

$$\frac{U}{L} = \frac{Q_{\text{produced}}}{L} \frac{Q_{\text{consumed}}}{Q_{\text{produced}}} (M_t)^{\frac{1}{\sigma-1}} * \\ * \left[\left(\frac{M_m}{M_t} \right)^{1-\rho} \left(\frac{Q_m}{Q_{\text{consumed}}} \frac{M_m (\int_z q_m^\rho(\varphi) \mu(\varphi) d\varphi)^{1/\rho}}{Q_m} \right)^\rho + \left(\frac{M}{M_t} \right)^{1-\rho} \left(\frac{Q_d}{Q_{\text{consumed}}} \frac{M (\int_x q^\rho(\varphi) \mu(\varphi) d\varphi)^{1/\rho}}{Q_d} \right)^\rho \right]^{1/\rho},$$

⁸The detailed derivation of this condition is given in the Appendix.

where $M_t = M + M_m$ is the total variety consumed at Home.

The first component in the product above is the productivity index in the economy measured as the total output per worker.⁹ It is important to note here that we are simply adding physical units of different goods to arrive at a concept of aggregate quantities consumed and produced.

The second component is the trade-adjusted terms of trade (TOT) index, which tells us the ratio of consumption to production in an open economy. To see this, we can rewrite it as

$$\begin{aligned} TOT &= \frac{Q_{consumed}}{Q_{produced}} = \frac{Q_d + Q_m}{Q_d + Q_{exp}} = \frac{Q_m}{Q_{exp}} \frac{Q_d + Q_m}{Q_m} \frac{Q_{exp}}{Q_d + Q_{exp}} \\ &= \frac{P_m Q_m}{P_{exp} Q_{exp}} \frac{P_{exp}}{P_m} \left[\frac{Q_d + Q_m}{Q_m} \frac{Q_{exp}}{Q_d + Q_{exp}} \right] = \frac{P_{exp}}{P_m} \left[\left(\frac{Q_{exp}}{Q_d + Q_{exp}} \right) / \left(\frac{Q_m}{Q_d + Q_m} \right) \right], \end{aligned} \quad (17)$$

where $P_{exp} = R_{exp}/Q_{exp}$ and $P_m = R_m/Q_m$. In other words, our TOT index takes into account “the importance of trade” in the economy: the ratio of the price of export to the price of import (the traditional terms of trade ratio) is multiplied by the ratio of the export share in production to the import share in consumption, which can be treated as a measure of the importance of the export relative to import in the economy. Note that if there is no trade, the TOT index equals 1.

The third component in the utility function is the familiar variety index, and the final component is the curvature term, which includes both within and cross country heterogeneity. To better understand it, first, note that since both countries have the same productivity distributions,

$$\frac{M(\int_x q^\rho(\varphi)\mu(\varphi)d\varphi)^{1/\rho}}{Q_d} = \frac{M_m(\int_z q_m^\rho(\varphi)\mu(\varphi)d\varphi)^{1/\rho}}{Q_m} = \frac{\beta - \sigma}{\beta} \left(\frac{\beta}{\beta - (\sigma - 1)} \right)^{\frac{1}{\rho}} < 1.$$

This term serves as a measure of heterogeneity among firms within each country. As shown in Appendix, it rises, if the dispersion of the productivity distribution falls, i.e., if β rises. And it converges to 1 as $\beta \rightarrow \infty$. In other words, for any value of σ , this term becomes closer to 1 as firms differ less, and it equals to 1, if all firms are identical. Moreover, if σ rises, then this term rises as well and becomes closer to 1, which reflects the fact that with higher elasticity of substitution, differences between varieties and their prices matter less. The curvature term can be rewritten as

$$\left[\frac{\beta - \sigma}{\beta} \left(\frac{\beta}{\beta - (\sigma - 1)} \right)^{\frac{1}{\rho}} \right] * \left[\left(\frac{M_m}{M_t} \right)^{1-\rho} \left(\frac{Q_m}{Q_{consumed}} \right)^\rho + \left(\frac{M}{M_t} \right)^{1-\rho} \left(\frac{Q_d}{Q_{consumed}} \right)^\rho \right]^{1/\rho}.$$

The second component in the expression above reflects cross country heterogeneity. To see this, note that since each (domestic or foreign) variety enters the utility function symmetrically, in the absence of any heterogeneity in prices within and across countries, households would consume the same quantity of each good so that this term would be equal to 1.

⁹This definition of productivity differs from that in Melitz (2003), who aims to capture “measured” productivity. In particular, he adds value added across firms and divides this sum by the industry level price, whereas we sum up value added across firms dividing by the price, or $(pq)/p = q$.

To sum up, there are 4 channels through which any policy affects welfare in the economy:

$$\frac{U}{L} = (\text{Productivity Index}) * (\text{TOT index}) * (\text{Variety Index}) * (\text{Curvature}). \quad (18)$$

3 The First Best Allocation

Now let us look at the social planner's choice of the optimal policy.¹⁰ The social planner chooses an allocation that maximizes (1) subject to full employment and balanced trade. Note that while he has a full control over domestic firms, with foreign firms he can only choose the level of expenditures on the imported goods. Then, as shown in Appendix, if there exists a solution to this problem, it is unique. Moreover, we can prove the following proposition:

Proposition 1 *The first best outcome can be achieved through implementing the following policies:*

- a consumption subsidy $1 - \eta$, where $\eta = \frac{\beta - \rho}{\beta} < 1$;
- an export tax $\tau = 1 - s$, where $s = \frac{\beta - \rho}{\beta} < 1$;
- an import tariff $t = \frac{\beta}{\beta - \rho} > 1$.

Proof. The detailed proof can be found in Appendix. Here we provide the sketch of this proof. First, we derive the system of the first order conditions (F.O.C.s) for the social planner's problem. We show that if there exist 2 solutions to it, then they coincide. Then we prove the sufficiency of F.O.C.s by looking at the matrix of the second derivatives evaluated at the solution point.

Next, for each policy, we obtain the market equilibrium conditions and derive the optimal values directly by showing that the first derivative of the utility function is negative for any values of policy parameter below the optimal one (for example, for all consumption subsidies with $\eta < (\beta - \rho) / \beta$), and it is positive for all values above the optimal one. Thus, the optimal value is the one when the first derivative equals to zero. Note that here we do not need the utility function be concave.

Finally, we show that depending on the choice of Lagrangian multipliers in the social planner's problem, the system of F.O.C.s coincides with the system of the market equilibrium conditions for each of all three policies mentioned in this proposition. ■

The intuition behind these result is the following. There are two distortions in the economy. First, there is a domestic distortion created by the mark-up: domestic goods are sold at a price above the opportunity cost, whereas imported goods are sold at a price equal to the opportunity cost, so in the equilibrium there is too little consumption of domestic relative to foreign varieties. This distortion is neutralized with the consumption subsidy, which allows consumers to pay a price equal to the producer's marginal cost ($\eta = \rho$). Another way to neutralize this distortion is to set an import tariff, which makes consumers pay the same "mark-up" $\frac{1}{\rho}$ while buying the imported

¹⁰Here we look at the social planner who maximizes welfare in the small economy only, not in the whole world.

variety as the one they pay for domestic varieties ($t = \frac{1}{\rho}$). Finally, by taxing export ($s = \rho$), the social planner makes exporting less attractive to producers, so that resources are shifted toward domestic production and the quantity of each consumed variety rises.

The second distortion in the model is caused by the decision of foreign producers to export, since they do not take into account a consumer surplus generated by their entry into the domestic market. Thus, the mass of the imported varieties M_m is below its optimal value. Such distortion can be neutralized by using policies opposite to those in the previous case: now the social planner needs a consumption tax or an export subsidy, $\eta = s = \frac{\beta - \rho}{\beta\rho}$, or an import subsidy, $t = \frac{\beta\rho}{\beta - \rho}$.¹¹

As a result, in the presence of both distortions in the economy, the optimal policy is a product of two policies needed to neutralize these distortions:

$$\eta = s = \rho \left(\frac{\beta - \rho}{\beta\rho} \right) = \frac{\beta - \rho}{\beta} < 1 \quad \text{and} \quad t = \frac{1}{\rho} \left(\frac{\beta\rho}{\beta - \rho} \right) = \frac{\beta}{\beta - \rho} > 1.$$

Note that in all cases the mark-up distortion dominates the consumer surplus distortion, so that the resulting policies are a consumption subsidy, an export tax and an import tariff.

To compare these policies with each other, note that while the “real” values, namely, cutoffs x , y , and z , and masses M , M_{exp} , and M_m , are the same in each case, the “nominal” values, namely, wage w , total revenues R , and price index P , can differ:

$$w^{cs} = \frac{\beta}{\beta - \rho} w^{\text{exp}} = w^m, \quad R^{cs} = R^{\text{exp}} = \frac{\beta}{\beta - \rho} R^m, \quad P^{cs} = P^{\text{exp}} = \frac{\beta}{\beta - \rho} P^m,$$

where “*cs*”, “*exp*”, and “*m*” denote the consumption subsidy, the export tax, and the import tariff cases, respectively.

First, note that the export tax leads to lower wage compared with the consumption subsidy, but the price index and total revenues are the same. The intuition is that the export tax reduces the demand for labor, since exporting is not that attractive option anymore, and as a result, wage is lower in this case. However, price indices are the same, since the prices of the imported varieties are still the same, and the price of any domestic variety is low in both cases either because of the consumption subsidy or lower wage in the export tax case. The revenues are the same since in one case the revenues from the export tax compensate for the low labor payments, and in the other case higher wage compensates for losses due to financing of the consumption subsidy.

Second, the wages are the same in the case of consumption subsidy and import tariff, however, the price index and revenues are higher in the latter case. The explanation of higher price index is that consumers have to pay a mark-up on both domestic and imported varieties. However, they have higher income due to the revenues from import tariff, and this income allows them to buy the same quantities of every variety as in the case of the consumption subsidy.

¹¹These optimal values can be derived if the mark-up distortion is neutralized by assuming perfect competition in the differentiated good sector. Details are available upon request.

In addition, Proposition 1 leads to the following straightforward conclusion:

Corollary 1 *In the presence of the optimal consumption subsidy, any trade policy results in welfare losses.*

4 The Effects of Export Subsidies

In this section, we assume that the government has in place the optimal consumption subsidy (i.e., $\eta = \frac{\beta-\rho}{\beta}$) and explore how export subsidies affect the four major components of the utility function in (18). Note that from Corollary 1, an introduction of the export subsidy worsens the equilibrium outcome compared to the case with no subsidy at all.¹² Moreover, we prove the following result:

Proposition 2 *An introduction of an export subsidy leads to the welfare losses: the higher is the subsidy, the lower is the welfare level.*

Proof. In Proposition 1 we already proved this result in the absence of consumption subsidy. Now the proof is the same except that the optimal value of subsidy is now $s = 1$. ■

To understand better why the increasing export subsidy causes a welfare reduction, we want to look at the components of the per capita utility function in (18). Before analyzing them, we first look at the effect of the export subsidy on the basic variables in the economy.

First, note that from the (FE) condition, the cutoffs for domestic producers and exporters, x and y , always move in the opposite directions in response to changes in the export subsidy s . Moreover, as we prove in Appendix, as s rises, y must fall, so that x must increase. In turn, from (11), M falls, $M_e = \frac{\delta M}{1-G(x)} = \frac{(\theta-1)L}{\sigma\theta f_e}$ remains constant, and $M_{\text{exp}} = M \frac{1-G(y)}{1-G(x)} = \frac{(\theta-1)b^\beta L}{\sigma\theta\delta f_e} y^{-\beta}$ rises. Finally, it can be shown that the wage increases with s , while z falls and M_m rises.

Proposition 3 *As the export subsidy increases, the productivity cutoff for domestic producers rises, the productivity cutoffs for exporters and importers fall, the wage rises, the mass of entrants remains unchanged, the mass of domestic producers falls, and the masses of exporters and importers increase.*

The intuition behind the results is that an increasing export subsidy allows less productive firms to export, so that the cutoff for exporters falls and their mass increases, which leads to the similar changes in the characteristics of importers in order to keep trade balance. At the same time, the demand for labor in the economy rises, which leads to a higher wage and makes it harder to produce for the domestic market, so that the cutoff for domestic producers rises and their mass falls. These two effects compensate each other so that there is no additional entry as a result.

¹²As also follows from Proposition 1, the export subsidy reduces welfare even in the absence of the consumption subsidy, since the optimal policy is the export tax.

Now let us look at the productivity index. From (14) and (16), it can be written as

$$\frac{Q_{produced}}{L} = \frac{Q_d + Q_{exp}}{L} = \frac{\beta(\sigma - 1)}{\beta - \sigma} M \left[\frac{f}{L} x + m_{exp} \frac{f_{exp}}{L} y \right].$$

Using the (M) and (FE) conditions, we can rewrite it as

$$\frac{Q_{produced}}{L} = \frac{\beta(\sigma - 1)}{\sigma\theta(\beta - \sigma)} \frac{\delta f_e}{(\theta - 1)b^\beta} \left[f + f_{exp} \left(\frac{x}{y} \right)^{\beta-1} \right] x^{\beta+1}.$$

Since $\beta > \sigma > 1$, and x and $\frac{x}{y}$ rise with s , the productivity index rises as well:

Proposition 4 *The productivity index is an increasing function of the export subsidy.*¹³

Intuitively, the increasing export subsidy raises the expected profits from exporting, thus, more firms enter the market. Competition becomes more severe and only the most productive firms can survive. As a result, labor is reallocated from less to more productive firms, and productivity increases, which is a standard selection effect. However, from Corollary 1 and Proposition 4 together, it is clear that in spite of this, welfare falls with the export subsidy because the other three components in (18) *together* fall and more than compensate for the productivity increase.

Now we want to look at the behavior of the TOT and variety indices separately. It can be shown numerically that depending on the parameters, each of TOT and variety indices can rise or fall with s .¹⁴ Thus, it is impossible to make unambiguous predictions about the behavior of these two indices in general. The intuition behind these results is the following. First, let us look at the TOT index. The export subsidy affects the terms of trade through two channels. The first is the intensive margin, i.e., the export subsidy allows the original exporters to increase the quantity they sell abroad, and this leads to the standard negative effect on the terms of trade.¹⁵ The second channel is along the extensive margin, as the export subsidy allows more firms to become exporters. As a result, the average productivity of exporters declines and this improves the TOT.

Now consider the variety index. Since the higher export subsidy results in the exit of the least efficient producers, the mass of domestic varieties falls. However, the amount of the imported variety rises. Thus, when the costs of exporting are very high and, as a result, the economy imports too little of the foreign variety, an increase in the imported variety can more than compensate for welfare losses due to a fall in the domestic variety, as consumers value an increase in the quantity of the imported variety a lot. As a result, the variety index can fall.

¹³Note that the productivity index also rises with the export subsidy in the absence of the consumption subsidy.

¹⁴The details can be found in the Appendix.

¹⁵It can be shown that the price set by the original exporters $p_{exp}(\varphi) = \frac{w}{s\rho\varphi}$. Since w/s falls with s , then this price falls as s rises.

5 Conclusion

Recent trade models with heterogenous firms suggest that export subsidies can indeed increase productivity by inducing a reallocation of labor from less to more productive firms. We have shown in this paper that, with an appropriate measure of productivity, this positive effect is in fact present, but is dominated by the negative effects of the export subsidy on the country's terms of trade and variety. The main message that arises from our results is that an exclusive focus on productivity can be counterproductive: a broader analysis is necessary.

We have also shown that the policy-makers have several options available to improve welfare in the economy with the mark-up and consumer surplus distortions: either the consumption subsidy or a small export tax, or a small import tariff can lead to the first best allocation in the economy.

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6 Appendix

6.1 Derivation of (z) condition (formula (13) in the paper).

Using (8) and (TB) condition in (7), we get

$$\begin{aligned} R &= \sigma F_{\text{exp}} \left(\frac{\gamma}{\rho z} \right)^{\sigma-1} P^{1-\sigma} = \sigma F_{\text{exp}} \left(\frac{\gamma}{\rho z} \right)^{\sigma-1} \left[\theta M_m \left(\frac{\rho z}{\gamma} \right)^{\sigma-1} + \theta M \left(\frac{\rho x}{\eta w} \right)^{\sigma-1} \right] \\ &= \sigma \theta M \left[F_{\text{exp}} \frac{M_m}{M} + F_{\text{exp}} \left(\frac{\gamma x}{\eta w z} \right)^{\sigma-1} \right] = \sigma \theta M \left[\frac{w}{s} m_{\text{exp}} f_{\text{exp}} + F_{\text{exp}} \left(\frac{\gamma x}{\eta w z} \right)^{\sigma-1} \right]. \end{aligned}$$

Finally, plugging (12) for R and dividing both parts by w , we get (z) condition

$$1 = \sigma \theta M \left[\frac{(s-1)}{s} f_{\text{exp}} m_{\text{exp}} + (1-\eta) f + \frac{1}{s} m_{\text{exp}} f_{\text{exp}} + \frac{F_{\text{exp}}}{w} \left(\frac{\gamma x}{\eta w z} \right)^{\sigma-1} \right]. \quad (19)$$

6.2 Curvature term

$$\frac{M(\int_x q^\rho(\varphi) \mu(\varphi) d\varphi)^{1/\rho}}{Q_d} = \frac{M \theta^{1/\rho} f(\sigma-1) x}{f(\sigma-1) \frac{\beta}{\beta-\sigma} M x} = \frac{\beta-\sigma}{\beta} \left(\frac{\beta}{\beta-(\sigma-1)} \right)^{\frac{\sigma}{\sigma-1}}. \quad (20)$$

Similarly, $M_m(\int_z q_m^\rho(\varphi) \mu(\varphi) d\varphi)^{1/\rho} / Q_m = \frac{\beta-\sigma}{\beta} \left(\frac{\beta}{\beta-(\sigma-1)} \right)^{\frac{1}{\rho}}$. In the case of the Pareto distribution, $G(\varphi) = 1 - \left(\frac{b}{\varphi}\right)^\beta$, $\varphi > b$, $E(\varphi) = \frac{\beta}{\beta-1} b$ and $Var(\varphi) = b^2 \frac{\beta}{(\beta-2)(\beta-1)^2}$. As a result, if β rises, the mean and dispersion fall, and if $\beta \rightarrow \infty$, then $E(\varphi) \rightarrow b$, and $Var(\varphi) \rightarrow 0$. In other words, an increase in β reduces heterogeneity among firms, and if $\beta \rightarrow \infty$, all firms are identical. What happens with (20) in both cases? It rises with β , since its derivative with respect to β is positive:

$$\frac{\sigma}{\beta^2} \left(\frac{\beta}{\beta-(\sigma-1)} \right)^{\frac{\sigma}{\sigma-1}} - \frac{(\beta-\sigma)\sigma}{\beta(\sigma-1)} \left(\frac{\beta}{\beta-(\sigma-1)} \right)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{[\beta-(\sigma-1)]^2} = \left(\frac{\beta}{\beta-(\sigma-1)} \right)^{\frac{\sigma}{\sigma-1}+1} \frac{\sigma}{\beta^3} > 0,$$

and it converges to 1 as $\beta \rightarrow \infty$. Moreover, if σ rises, it falls (given that $\sigma < \beta$): $\frac{\beta-\sigma}{\beta} \left(\frac{\beta}{\beta-(\sigma-1)} \right)^{\frac{\sigma}{\sigma-1}} = \left(1 - \frac{1}{\beta-(\sigma-1)} \right) \left(\frac{\beta}{\beta-(\sigma-1)} \right)^{\frac{1}{\sigma-1}}$, where the first part falls. The second part falls as well, since $\frac{1}{(\sigma-1)} \ln \left(\frac{\beta}{\beta-(\sigma-1)} \right)$ falls with σ . What happens if $\sigma \rightarrow \infty$? In our model, we have a restriction $\beta > \sigma$. Thus, σ is always bounded from above. And if $\sigma \rightarrow \infty$, it means that $\beta \rightarrow \infty$, so that this term $\rightarrow 1$.

6.3 Proof of Proposition 1

6.3.1 Social Planner' Problem and Its Solution.

Let $q(\varphi)$ be the quantity consumed of a good with productivity index φ and let $Q(\varphi)$ be the quantity produced. Then if all varieties $v \in \Omega$ are produced, it must be that $q(v)$ with $v \in \Omega$ maximizes utility

$\int_{v \in \Omega} q(v)^\rho dv$ s.t. $\int_{v \in \Omega} [q(v)/\varphi(v)] dv = K$. This leads to the F.O.C. of $q(v)/q(v') = [\varphi(v)/\varphi(v')]^\sigma$. On the other hand, if all varieties $v \in \Omega$ are exported, then it must be that $Q(v) - q(v)$ maximizes export revenue $\int_{v \in \Omega} a(Q(v) - q(v))^\rho dv$ s.t. $\int_{v \in \Omega} [(Q(v) - q(v))/\varphi(v)] dv = J$. This leads to the F.O.C. $\frac{Q(v) - q(v)}{Q(v') - q(v')} = [\varphi(v)/\varphi(v')]^\sigma$. Combining both results, we obtain $Q(v)/Q(v') = [\varphi(v)/\varphi(v')]^\sigma$. Thus, an optimal allocation would necessarily have $q(\varphi) = \phi\varphi^\sigma$ and $Q(\varphi) - q(\varphi) = \alpha\varphi^\sigma$, with $\alpha, \phi > 0$ (for the appropriate levels of φ). Moreover, if a variety v with $\varphi(v)$ is consumed (exported), then all varieties with $\varphi > \varphi(v)$ must be consumed (exported).

In addition, if all imported varieties $v \in \Omega_m$ are consumed, then it must be that $q_m(v)$ with $v \in \Omega_m$ maximizes utility $\int_{v \in \Omega_m} q_m(v)^\rho dv$ s.t. $\int_{v \in \Omega_m} p_m(v) q_m(v) dv = K_m$, where K_m is chosen by social planner. Then $q_m(v)/q_m(v') = [p_m(v')/p_m(v)]^\sigma$, and an importer with productivity φ sets a price $p_m(\varphi) = \frac{\gamma}{\rho\varphi}$, where γ denotes the marginal costs of production abroad, so that $q_m(v)/q_m(v') = [\varphi(v)/\varphi(v')]^\sigma$. Assume $q_m(\varphi) = i\varphi^\sigma$. If a variety v with $\varphi(v)$ is imported, then all varieties with $\varphi > \varphi(v)$ must be imported. Let us denote the lowest productivity of the importers by z , then it has to satisfy the zero profit condition:

$$iz^\sigma \frac{\gamma}{\rho z} = \sigma F_{\text{exp}}, \quad \text{or} \quad i = \frac{(\sigma - 1) F_{\text{exp}}}{\gamma z^{\sigma-1}}.$$

Thus, we look for an allocation that maximizes welfare, has no goods produced for $\varphi < x$, exports only for goods with $\varphi > y$, subject to full employment and balanced trade:

$$\begin{aligned} & \max_{x,y,z,M} \left\{ \int_z^\infty q_m(\varphi)^\rho M_m \mu(\varphi) d\varphi + \int_x^\infty q(\varphi)^\rho M \mu(\varphi) d\varphi \right\} \quad \text{s.t.} \\ & \int_x^\infty (f + q(\varphi)/\varphi) M \mu(\varphi) d\varphi + \int_y^\infty \left(f_{\text{exp}} + \frac{Q(\varphi) - q(\varphi)}{\varphi} \right) M \mu(\varphi) d\varphi + \frac{M \delta f_e}{1 - G(x)} = 1, \\ & \int_y^\infty p_{\text{exp}} (Q(\varphi) - q(\varphi)) M \mu(\varphi) d\varphi = \int_z^\infty p_m q_m(\varphi) M_m \mu(\varphi) d\varphi, \end{aligned}$$

where $M_m = 1 - G(z)$. In addition, export revenues are $(Q - q)p_{\text{exp}}$. But $Q - q = A p_{\text{exp}}^{-\sigma}$ implies $a(Q - q)^{-1/\sigma} = p_{\text{exp}}$, where $a = A^{1/\sigma}$. Hence, export revenues are $a(Q - q)^{1-1/\sigma} = a(Q - q)^\rho$. Also, recall that we assume the Pareto productivity distribution, $G(\varphi) = 1 - \left(\frac{b}{\varphi}\right)^\beta$. Thus, we have

$$\max_{x,y,z,M,\alpha,\phi} \left\{ M_m v^\rho \theta z^{\sigma-1} + M \phi^\rho \theta x^{\sigma-1} \right\} \quad \text{s.t.}$$

$$M \left[f + \theta \phi x^{\sigma-1} + m_{\text{exp}} f_{\text{exp}} + m_{\text{exp}} \theta \alpha y^{\sigma-1} + \frac{\delta f_e}{b^\beta} x^\beta \right] = 1, \quad \text{and} \quad M_m v^\rho z^{\sigma-1} = M m_{\text{exp}} a \alpha^\rho y^{\sigma-1},$$

where $m_{\text{exp}} = (1 - G(y)) / (1 - G(x)) = (x/y)^\beta$. We can rewrite it as

$$\max_{x,y,z,M,\alpha,\phi} \left\{ \left(\frac{(\sigma - 1) F_{\text{exp}}}{\gamma} \right)^\rho b^\beta z^{\rho-\beta} + M \phi^\rho x^{\sigma-1} \right\} \quad \text{s.t.}$$

$$M \left[f + \theta \phi x^{\sigma-1} + m_{\text{exp}} f_{\text{exp}} + m_{\text{exp}} \theta \alpha y^{\sigma-1} + \frac{\delta f_e}{b^\beta} x^\beta \right] = 1, \quad \text{and} \quad \sigma F_{\text{exp}} b^\beta z^{-\beta} = M m_{\text{exp}} a \alpha^\rho y^{\sigma-1},$$

The Lagrangian is then:

$$\begin{aligned} \mathcal{L} = & \left(\frac{(\sigma-1) F_{\text{exp}}}{\gamma} \right)^\rho b^\beta z^{\rho-\beta} + M \phi^\rho x^{\sigma-1} + \zeta \left(M m_{\text{exp}} a \alpha^\rho y^{\sigma-1} - \sigma F_{\text{exp}} b^\beta z^{-\beta} \right) \\ & - \lambda \left(M f + M \theta \phi x^{\sigma-1} + M m_{\text{exp}} f_{\text{exp}} + M m_{\text{exp}} \theta \alpha y^{\sigma-1} + \frac{M \delta f_e}{b^\beta} x^\beta - 1 \right). \end{aligned} \quad (21)$$

This must be maximized with respect to z, x, y, α, M . Letting $h(v) = g(v)/[1 - G(v)]$, then:

$$(z) : \quad \partial \mathcal{L} / \partial z = (\rho - \beta) \left(\frac{(\sigma-1) F_{\text{exp}}}{\gamma} \right)^\rho b^\beta z^{\rho-\beta-1} + \zeta \beta \sigma F_{\text{exp}} b^\beta z^{-\beta-1} = 0, \quad (22)$$

$$\begin{aligned} (x) : \quad \partial \mathcal{L} / \partial x = & M \phi^\rho \theta (\sigma-1) x^{\sigma-2} - \lambda M (\sigma-1) \theta \phi x^{\sigma-2} - \lambda M m_{\text{exp}} f_{\text{exp}} h(x) - \lambda M m_{\text{exp}} h(x) \theta \alpha y^{\sigma-1} \\ & - \lambda M \delta f_e \frac{h(x)}{1-G(x)} + \zeta M m_{\text{exp}} h(x) a \alpha^\rho \theta y^{\sigma-1} = 0, \end{aligned} \quad (23)$$

$$\begin{aligned} (y) : \quad \partial \mathcal{L} / \partial y = & \lambda M f_{\text{exp}} m_{\text{exp}} h(y) + \lambda M m_{\text{exp}} h(y) \alpha \theta y^{\sigma-1} - \lambda M m_{\text{exp}} \alpha \theta (\sigma-1) y^{\sigma-2} \\ & - \zeta M m_{\text{exp}} h(y) a \alpha^\rho \theta y^{\sigma-1} + \zeta M m_{\text{exp}} a \alpha^\rho \theta (\sigma-1) y^{\sigma-2} = 0, \end{aligned} \quad (24)$$

$$(\phi) : \quad \partial \mathcal{L} / \partial \phi = M \rho \phi^{\rho-1} \theta x^{\sigma-1} - \lambda M \theta x^{\sigma-1} = 0, \quad (25)$$

$$(\alpha) : \quad \partial \mathcal{L} / \partial \alpha = -\lambda M m_{\text{exp}} \theta y^{\sigma-1} + \zeta M m_{\text{exp}} a \rho \alpha^{\rho-1} \theta y^{\sigma-1} = 0, \quad (26)$$

$$(M) : \quad \partial \mathcal{L} / \partial M = \phi^\rho \theta x^{\sigma-1} - \lambda f - \lambda \theta \phi x^{\sigma-1} - \lambda f_{\text{exp}} m_{\text{exp}} - \lambda m_{\text{exp}} \alpha \theta y^{\sigma-1} - \lambda \frac{\delta f_e}{1-G(x)} + \zeta m_{\text{exp}} a \alpha^\rho \theta y^{\sigma-1} = 0. \quad (27)$$

Note that $\theta(\sigma-1)v^{\sigma-2} = (\theta-1)h(v)v^{\sigma-1}$, hence, we have 8 equations with 8 unknown variables:

$$(z) : \quad \frac{\beta - \rho}{\beta} z^\rho \left(\frac{(\sigma-1) F_{\text{exp}}}{\gamma} \right)^\rho = \zeta \sigma F_{\text{exp}},$$

$$(x) : \quad \phi (\theta-1) x^{\sigma-1} (\phi^{\rho-1} - \lambda) - \lambda m_{\text{exp}} f_{\text{exp}} - \lambda m_{\text{exp}} \theta \alpha y^{\sigma-1} - \lambda \delta f_e \frac{1}{1-G(x)} + \zeta m_{\text{exp}} a \alpha^\rho \theta y^{\sigma-1} = 0,$$

$$\begin{aligned} (y) : \quad & \lambda f_{\text{exp}} + \lambda \alpha \theta y^{\sigma-1} - \lambda \alpha (\theta-1) y^{\sigma-1} - \zeta a \alpha^\rho \theta y^{\sigma-1} + \zeta a \alpha^\rho (\theta-1) y^{\sigma-1} \\ & = \lambda f_{\text{exp}} + \lambda \alpha y^{\sigma-1} - \zeta a \alpha^\rho y^{\sigma-1} = 0, \end{aligned}$$

$$(\phi) : \quad \rho \phi^{\rho-1} = \lambda,$$

$$(\alpha) : \quad \zeta a \rho \alpha^{\rho-1} = \lambda,$$

$$(M) : \quad \phi^\rho \theta x^{\sigma-1} - \lambda f - \lambda \theta \phi x^{\sigma-1} - \lambda f_{\text{exp}} m_{\text{exp}} - \lambda m_{\text{exp}} \alpha \theta y^{\sigma-1} - \lambda \frac{\delta f_e}{1-G(x)} + \zeta m_{\text{exp}} a \alpha^\rho \theta y^{\sigma-1} = 0,$$

$$(FE) : \quad 1 = M f + M \theta \phi x^{\sigma-1} + M f_{\text{exp}} m_{\text{exp}} + M m_{\text{exp}} \alpha \theta y^{\sigma-1} + \frac{M \delta f_e}{1-G(x)},$$

$$(TB) : \quad \sigma F_{\text{exp}} b^\beta z^{-\beta} = M m_{\text{exp}} a \alpha^\rho y^{\sigma-1}.$$

Moreover, subtracting (M) from (x) gives:

$$-\phi^\rho x^{\sigma-1} + \lambda \phi x^{\sigma-1} + \lambda f = 0, \quad \text{or} \quad \lambda \phi x^{\sigma-1} - \phi^\rho x^{\sigma-1} = \lambda f.$$

From (ϕ), $\lambda = \rho \phi^{\rho-1}$. Thus, (x) is $\phi x^{\sigma-1} = \sigma \rho f$. Similarly, using (α) in (y) gives $\alpha y^{\sigma-1} = \sigma \rho f_{\text{exp}}$. Using new equations (x), (y), (ϕ), and (α), we derive new (M) and (FE) conditions:

$$\frac{\delta f_e}{1 - G(x)} = \theta (\sigma \rho f) \left(\frac{1}{\rho} - 1 \right) - f - f_{\text{exp}} m_{\text{exp}} + \theta m_{\text{exp}} (\sigma \rho f_{\text{exp}}) \left(\frac{1}{\rho} - 1 \right) = (\theta - 1) (f + f m_{\text{exp}}),$$

$$1 = M [f + \theta \sigma \rho f + f_{\text{exp}} m_{\text{exp}} + m_{\text{exp}} \theta \sigma \rho f_{\text{exp}} + (\theta - 1) (f + f m_{\text{exp}})] = \sigma \theta M (f + m_{\text{exp}} f_{\text{exp}}).$$

Thus, we have the following **system of F.O.C.s in the social planner's problem**:

$$(z) : \quad \frac{\beta - \rho}{\beta} z^\rho \left(\frac{(\sigma - 1) F_{\text{exp}}}{\gamma} \right)^\rho = \zeta \sigma F_{\text{exp}}, \quad (28)$$

$$(x) : \quad \phi x^{\sigma-1} = \sigma \rho f, \quad (29)$$

$$(y) : \quad \alpha y^{\sigma-1} = \sigma \rho f_{\text{exp}}, \quad (30)$$

$$(\phi) : \quad \rho \phi^{\rho-1} = \lambda, \quad (31)$$

$$(\alpha) : \quad \zeta a \rho \alpha^{\rho-1} = \lambda, \quad (32)$$

$$(M) : \quad \frac{\delta f_e}{1 - G(x)} = (\theta - 1) (f + m_{\text{exp}} f_{\text{exp}}), \quad (33)$$

$$(FE) : \quad 1 = \sigma \theta M (f + m_{\text{exp}} f_{\text{exp}}), \quad (34)$$

$$(TB) : \quad \sigma F_{\text{exp}} b^\beta z^{-\beta} = M m_{\text{exp}} a \alpha^\rho y^{\sigma-1}. \quad (35)$$

Uniqueness of the Solution It can be shown that if there are 2 solutions, and both solutions have at least one common component (for example, $x_1 = x_2$), then these solutions coincide. We will prove that there should be a unique x , which solves the system, thus, if the solution exists, it is unique. To do this, we will rewrite the system above till we have 1 equation with 1 unknown variable, which has a unique solution. First, let us exclude M . From (TB), $M = \sigma F_{\text{exp}} b^\beta z^{-\beta} / (m_{\text{exp}} a \alpha^\rho \theta y^{\sigma-1})$. M is used only in (FE), which together with (y) and (α) can be written as :

$$\begin{aligned} (FE) : \quad 1 &= \sigma \theta \frac{\sigma F_{\text{exp}} b^\beta z^{-\beta} (f + m_{\text{exp}} f_{\text{exp}})}{m_{\text{exp}} a \alpha^\rho y^{\sigma-1}} = \sigma \theta \frac{F_{\text{exp}} b^\beta z^{-\beta} (f + m_{\text{exp}} f_{\text{exp}})}{m_{\text{exp}} a \alpha^{\rho-1} \rho f_{\text{exp}}} \\ &= \sigma \theta \frac{F_{\text{exp}} b^\beta z^{-\beta} (f + m_{\text{exp}} f_{\text{exp}})}{m_{\text{exp}} f_{\text{exp}} \lambda / \zeta} = \frac{\zeta}{\lambda} \sigma \theta \frac{F_{\text{exp}} b^\beta z^{-\beta} (f + m_{\text{exp}} f_{\text{exp}})}{m_{\text{exp}} f_{\text{exp}}}. \end{aligned}$$

Let us exclude λ and ζ . From (z) and (ϕ) : $\zeta = \frac{1}{\sigma F_{\text{exp}}} \frac{\beta - \rho}{\beta} z^\rho \left(\frac{(\sigma - 1) F_{\text{exp}}}{\gamma} \right)^\rho$ and $\lambda = \rho \phi^{\rho - 1}$. Then,

$$\begin{aligned} (x) & : \quad \phi x^{\sigma - 1} = (\sigma - 1) f; & (y) & : \quad \alpha y^{\sigma - 1} = (\sigma - 1) f_{\text{exp}}; \\ (\alpha) & : \quad z^{\rho - 1} a \rho \alpha^{\rho - 1} = \phi^{\rho - 1}; & (M) & : \quad \frac{\delta f_e}{1 - G(x)} = (\theta - 1) (f + m_{\text{exp}} f_{\text{exp}}); \\ (FE) & : \quad 1 = \frac{\beta - \rho}{\rho \beta} \left(\frac{z (\sigma - 1) F_{\text{exp}}}{\gamma} \right)^\rho \frac{1}{\phi^{\rho - 1}} \theta \left(\frac{b}{z} \right)^\beta \frac{(f + m_{\text{exp}} f_{\text{exp}})}{m_{\text{exp}} f_{\text{exp}}}. \end{aligned}$$

Let us exclude z . From (α) , (x) , and (y) , $z = (a\rho)^{\frac{1}{\rho - 1}} \frac{\phi}{\alpha} = (a\rho)^{\frac{1}{\rho - 1}} \frac{f}{f_{\text{exp}}} \left(\frac{y}{x} \right)^{\sigma - 1}$, so we have now

$$\begin{aligned} (x) & : \quad \phi x^{\sigma - 1} = (\sigma - 1) f, & (y) & : \quad \alpha y^{\sigma - 1} = (\sigma - 1) f_{\text{exp}}, \\ (M) & : \quad \frac{\delta f_e}{1 - G(x)} = (\theta - 1) (f + m_{\text{exp}} f_{\text{exp}}), \\ (FE) & : \quad \text{Some constant} = \alpha^{1 - \rho} (m_{\text{exp}})^{\frac{(\sigma - 1)(1 - \beta)}{\beta}} \frac{f + m_{\text{exp}} f_{\text{exp}}}{f_{\text{exp}} m_x}. \end{aligned}$$

Let us exclude ϕ and α . From (x) and (y) : $\phi = \frac{(\sigma - 1) f}{x^{\sigma - 1}}$ and $\alpha = \frac{(\sigma - 1) f_{\text{exp}}}{y^{\sigma - 1}}$. Thus,

$$\begin{aligned} (M) & : \quad x = \left(b^\beta \frac{(\theta - 1)}{\delta f_e} (f + m_{\text{exp}} f_{\text{exp}}) \right)^{1/\beta}, \\ (FE) & : \quad \text{Some constant} = y^{-\rho} (m_{\text{exp}})^{\frac{(\sigma - 1)(1 - \beta)}{\beta}} \frac{f + m_{\text{exp}} f_{\text{exp}}}{f_{\text{exp}} m_x}, \text{ or} \\ (FE) & : \quad \text{Some constant} = (m_{\text{exp}})^{\frac{(\sigma - 1)(1 - \beta)}{\beta}} \left[\frac{f + m_{\text{exp}} f_{\text{exp}}}{f_{\text{exp}} m_x} \right]^{1 - \frac{\rho}{\beta}}. \end{aligned}$$

Note that equation (FE) can be rewritten as

$$(m_{\text{exp}})^{\frac{(\sigma - 1)(1 - \beta)}{\beta}} \left(\frac{f}{m_{\text{exp}}} + f_{\text{exp}} \right)^{1 - \frac{\rho}{\beta}} = \text{Some exogenously given constant.}$$

Then since $\beta > \sigma > 1 > \rho$, the left-hand side of equation above is a decreasing function of m_{exp} . Thus, the equation above has a unique solution m_{exp} . But from (M), it follows, that x is also unique! Therefore, we proved that if the solution of the system of F.O.C.s exists, it is unique.

Sufficiency of F.O.C.s Let us rewrite the Lagrangian as

$$\mathcal{L} = U + H, \text{ where } U = \left(\frac{(\sigma - 1) F_{\text{exp}}}{\gamma} \right)^\rho b^\beta z^{\rho - \beta} + M \phi^\rho \theta x^{\sigma - 1} \text{ and } H = (-\lambda, \zeta) \vec{h},$$

where \vec{h} is a vector of restrictions in our problem:

$$\vec{h} = \begin{pmatrix} Mf + M\theta\phi x^{\sigma-1} + Mm_{\text{exp}}f_{\text{exp}} + Mm_{\text{exp}}\theta\alpha y^{\sigma-1} + \frac{M\delta f_e}{b^\beta}x^\beta - 1 \\ Mm_{\text{exp}}a\alpha^\rho y^{\sigma-1} - \sigma F_{\text{exp}}b^\beta z^{-\beta} \end{pmatrix}.$$

To prove the sufficiency of the first order conditions of the social planner's problem described above, we need to show that for any vector $\vec{\chi}$ such that¹⁶

$$\vec{\chi} \neq \vec{0} \quad \text{and} \quad \nabla \vec{h}(\text{solution}) \vec{\chi} = \vec{0}, \quad (36)$$

we have

$$\vec{\chi}' \mathcal{L}_{\xi\xi}(\xi^*) \vec{\chi} < 0,$$

where $\mathcal{L}_{\xi\xi}(\xi^*)$ is the matrix of second derivatives of the Lagrangian with respect to $\xi' = (x, y, z, M, \alpha, \phi)$, evaluated at the solution point ξ^* . (See, for example, Theorem 3.3.2, p. 214 in G. Giorgi, A. Gueraggio and J. Thierfelder, "Mathematics of Optimization: Smooth and Nonsmooth Case", Elsevier B.V., 2004, which states that if there exist such λ and ζ , for which the conditions above are satisfied, then the solution we found is a point of global maximum of the objective function subject to our restrictions. And we found such λ and ζ already.) In order to do this, we can show that the matrix of the second derivatives can be written as (see the proof below):

$$\mathcal{L}_{\xi\xi}(\xi^*) = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a_{ii} < 0$ for any $i \neq 6$. Thus, $\vec{\chi}' \mathcal{L}_{\xi\xi}(\xi^*) \vec{\chi} = \sum_{i=1}^5 a_{ii} \chi_i^2 \leq 0$, so we need to show that $\vec{\chi}' \mathcal{L}_{\xi\xi}(\xi^*) \vec{\chi} \neq 0$. Note that the only way $\vec{\chi}' \mathcal{L}_{\xi\xi}(\xi^*) \vec{\chi} = 0$ is if $\vec{\chi}' = (0, 0, 0, 0, 0, \chi_6)$ and $\chi_6 \neq 0$. However, in this case (36) is violated since $\nabla \vec{h}(\text{solution}) \vec{\chi} = \vec{0}$ implies

$$\begin{aligned} \left[\frac{\partial}{\partial M} \left(Mf + M\theta\phi x^{\sigma-1} + Mm_{\text{exp}}f_{\text{exp}} + Mm_{\text{exp}}\theta\alpha y^{\sigma-1} + \frac{M\delta f_e}{b^\beta}x^\beta - 1 \right) \right] \chi_6 &= 0, \\ \left[\frac{\partial}{\partial M} \left(Mm_{\text{exp}}a\alpha^\rho y^{\sigma-1} - \sigma F_{\text{exp}}b^\beta z^{-\beta} \right) \right] \chi_6 &= 0, \quad \text{or} \end{aligned}$$

$$\left(f + \theta\phi x^{\sigma-1} + m_{\text{exp}}f_{\text{exp}} + m_{\text{exp}}\theta\alpha y^{\sigma-1} + \frac{\delta f_e}{b^\beta}x^\beta \right) \chi_6 = 0, \quad \text{and} \quad (m_{\text{exp}}a\alpha^\rho y^{\sigma-1}) \chi_6 = 0,$$

which is clearly impossible, since in the second equation above, $m_{\text{exp}}a\alpha^\rho\theta y^{\sigma-1} \neq 0$ and $\chi_6 \neq 0$.

¹⁶In the expression below, $\nabla \vec{h}(\text{solution})$ is a matrix of the first derivatives of the vector of restrictions in our problem with respect to $\xi' = (z, x, y, \phi, \alpha, M)$, evaluated at the solution point ξ^* .

The derivation of $\mathcal{L}_{\xi\xi}(\xi^*)$. Let denote the elements of $\mathcal{L}_{\xi\xi}(\xi^*)$ by $[a_{ij}]_{i,j=1,\dots,6}$.

Diagonal elements. First, note that

$$\begin{aligned} a_{11} &= \frac{\partial^2 \mathcal{L}}{\partial z^2} = (\rho - \beta)(\rho - \beta - 1) \left(\frac{(\sigma - 1) F_{\text{exp}}}{\gamma} \right)^\rho b^\beta z^{\rho - \beta - 2} - \zeta \beta (\beta + 1) \sigma F_{\text{exp}} b^\beta z^{-\beta - 2} \\ &= \zeta \sigma F_{\text{exp}} b^\beta z^{-\beta - 2} \beta (-\rho) < 0, \end{aligned}$$

where the second equality follows from equation (z).

What we do in cases $i = 2, 3$ is we take the derivatives and use the property that $\frac{\partial(x^n)}{\partial x} = \frac{n}{x}(x^n)$. Then we use the corresponding condition to simplify the expression for the derivative and compare it with 0. For example, since $\partial(m_{\text{exp}}h(x))/\partial x = \partial(\beta x^{\beta-1}/y^\beta)/\partial x = \frac{\beta-1}{x}m_{\text{exp}}h(x)$ and $\partial\left(\frac{h(x)}{1-G(x)}\right)/\partial x = \partial(\beta x^{\beta-1})/\partial x = \frac{\beta-1}{x}\frac{h(x)}{1-G(x)}$,

$$\begin{aligned} a_{22} &= \frac{\partial^2 \mathcal{L}}{\partial x^2} = M\phi^\rho \theta (\sigma - 1)(\sigma - 2)x^{\sigma-3} - \lambda\phi\theta (\sigma - 1)(\sigma - 2)x^{\sigma-3} \\ &\quad - \frac{\beta - 1}{x} \left(\lambda M m_{\text{exp}} f_{\text{exp}} h(x) - \lambda M m_{\text{exp}} h(x) \theta \alpha y^{\sigma-1} \lambda M \delta f_e \frac{h(x)}{1-G(x)} + \zeta M m_{\text{exp}} h(x) a \alpha^\rho \theta y^{\sigma-1} \right). \end{aligned}$$

We can use condition (x) to rewrite it as

$$\begin{aligned} a_{22} &= M\phi^\rho \theta (\sigma - 1)(\sigma - 2)x^{\sigma-3} - \lambda\phi\theta (\sigma - 1)(\sigma - 2)x^{\sigma-3} - \frac{\beta - 1}{x} [\phi^\rho \theta (\sigma - 1)x^{\sigma-2} - \lambda\phi\theta (\sigma - 1)x^{\sigma-2}] \\ &= M\phi\theta (\sigma - 1)x^{\sigma-3} [\phi^{\rho-1} - \lambda] [(\sigma - 1) - \beta]. \end{aligned}$$

Note that since $\rho\phi^{\rho-1} = \lambda$ and $\rho < 1$, then $\phi^{\rho-1} - \lambda > 0$, while $(\sigma - 1) - \beta < 0$. Thus, $a_{22} < 0$. Similarly, it can be shown that $a_{33} = \frac{\partial^2 \mathcal{L}}{\partial y^2} = -(\sigma - 1)M\lambda f_{\text{exp}} m_{\text{exp}} h(y) \frac{1}{y} < 0$, $a_{44} = \frac{\partial^2 \mathcal{L}}{\partial \phi^2} = (\rho - 1)\phi^{\rho-2}M\rho\theta x^{\sigma-1} < 0$, and $a_{55} = \frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = (\rho - 1)\zeta M m_{\text{exp}} a \rho \alpha^{\rho-2} \theta y^{\sigma-1} < 0$, while $a_{66} = \frac{\partial^2 \mathcal{L}}{\partial M^2} = \frac{\partial}{\partial M} \left(\frac{\partial \mathcal{L}}{\partial M} \right) = 0$.

Off-Diagonal elements. To derive the off diagonal elements, we use Young's theorem. As a result,

$$a_{12} = a_{21} = \frac{\partial^2 \mathcal{L}}{\partial z \partial x} = \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial z} \right) = \frac{\partial}{\partial x} (\rho z^{\rho-1} - \zeta) = 0, \quad \text{and similarly,}$$

$$a_{13} = a_{31} = a_{14} = a_{41} = a_{15} = a_{51} = a_{16} = a_{61} = 0.$$

$$a_{23} = a_{32} = \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial y} \right) = \frac{\beta}{x} \left(\frac{\partial \mathcal{L}}{\partial y} \right) = 0, \quad \text{as } \frac{\partial m_{\text{exp}}}{\partial x} = \frac{\beta}{x} m_{\text{exp}},$$

$$a_{24} = a_{42} = \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) = \frac{\sigma - 1}{x} \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) = 0, \quad a_{25} = a_{52} = \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \alpha} \right) = \frac{\beta}{x} \left(\frac{\partial \mathcal{L}}{\partial \alpha} \right) = 0,$$

$$a_{26} = a_{62} = \frac{\partial^2 \mathcal{L}}{\partial M \partial x} = \frac{\partial}{\partial M} \left(\frac{\partial \mathcal{L}}{\partial x} \right) = \frac{1}{M} \left(\frac{\partial \mathcal{L}}{\partial x} \right) = 0.$$

Using the same logic, it can be shown that all off-diagonal elements of the matrix are zeros.

6.3.2 The Consumption Subsidy

The Optimal Value of Consumption Subsidy. We need to rewrite the equilibrium conditions derived in Section 2 by setting $s = 1$:

$$\text{(EXP) condition} \quad A \left(\frac{\rho y}{w} \right)^{\sigma-1} = \sigma w f_{\text{exp}}, \quad (37)$$

$$\text{(FE) condition} \quad (\theta - 1) x^{-\beta} [f + m_{\text{exp}} f_{\text{exp}}] = \frac{\delta f_e}{b^\beta}, \quad (38)$$

$$\text{(TB) condition} \quad M_m F_{\text{exp}} = w M_{\text{exp}} f_{\text{exp}}, \quad (39)$$

$$\text{(M) condition} \quad M = \frac{1}{\sigma \theta (f + m_{\text{exp}} f_{\text{exp}})} = \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} x^{-\beta}. \quad (40)$$

$$\text{(z) condition} \quad 1 = \sigma \theta M \left[f_{\text{exp}} m_{\text{exp}} + (1 - \eta) f + \frac{F_{\text{exp}}}{w} \left(\frac{\gamma x}{\eta w z} \right)^{\sigma-1} \right] \quad (41)$$

Now we are ready to prove that **a consumption subsidy equal to $1 - \eta$, where $\eta = \frac{\beta - \rho}{\beta}$, results in the maximal level of welfare.**

Proof. Step 1. First, we prove that when η rises, y and w must fall and x must rise. From

$$\text{(EXP):} \quad w^\sigma = \frac{A \rho^{\sigma-1}}{\sigma f_{\text{exp}}} y^{\sigma-1}, \quad (42)$$

y and w must move in the same direction, and from the (FE) condition, x and y move in the opposite direction. Next,

$$\text{(z) + (M)} \quad \Rightarrow \quad 1 = \frac{1}{(f + m_{\text{exp}} f_{\text{exp}})} \left[f_{\text{exp}} m_{\text{exp}} + (1 - \eta) f + \frac{F_{\text{exp}}}{w} \left(\frac{\gamma x}{\eta w z} \right)^{\sigma-1} \right], \quad \text{or}$$

$$\frac{F_{\text{exp}}}{w} \left(\frac{\gamma x}{\eta w z} \right)^{\sigma-1} = \eta f, \quad \text{or} \quad (43)$$

$$w^\sigma = \frac{1}{\eta^\sigma} \frac{F_{\text{exp}}}{f} \left(\frac{\gamma x}{z} \right)^{\sigma-1}, \quad (44)$$

where from (TB) condition, $z^{-1} = \left(\frac{f_{\text{exp}}}{F_{\text{exp}}} w M_{\text{exp}} \right)^{\frac{1}{\beta}} \propto w^{\frac{1}{\beta}} y^{-1}$ so that

$$w^{\sigma - \frac{\sigma-1}{\beta}} \propto \frac{1}{\eta^\sigma} \left(\frac{x}{y} \right)^{\sigma-1}. \quad (45)$$

Now, assume that if η rises, then y rises and x falls. Then from (42), $w \propto y^\rho$ must rise, but

$$\text{from (45): } w \propto \left[\frac{1}{\eta^\sigma} \left(\frac{x}{y} \right)^{\sigma-1} \right]^{\frac{1}{\sigma-\frac{\sigma-1}{\beta}}} \text{ must fall (since } \beta > \sigma \text{).}$$

Our assumption led to the contradiction, thus, as η rises, y and w must fall and x must increase.

Step 2. Note that from (z) condition,

$$\int_z^\infty [q_m(\varphi)]^\rho M_m \mu(\varphi) d\varphi = M_m \left[\frac{F_{\text{exp}}}{\gamma} (\sigma - 1) \right]^\rho z^{\beta-\rho(\sigma-1)} \beta \int_z^\infty \varphi^{\rho\sigma-\beta-1} d\varphi = M_m \theta \left[\frac{F_{\text{exp}}}{\gamma} (\sigma - 1) z \right]^\rho,$$

$$\text{and } \int_x^\infty [q(\varphi)]^\rho M \mu(\varphi) d\varphi = M [f(\sigma - 1)]^\rho x^{\beta-\rho(\sigma-1)} \beta \int_x^\infty \varphi^{\rho\sigma-\beta-1} d\varphi = M \theta [f(\sigma - 1) x]^\rho.$$

We want to rewrite

$$U^\rho = \int_z^\infty [q_m(\varphi)]^\rho M_m \mu(\varphi) d\varphi + \int_x^\infty [q(\varphi)]^\rho M \mu(\varphi) d\varphi = \theta (\sigma - 1)^\rho \left[M_m \left(\frac{F_{\text{exp}}}{\gamma} z \right)^\rho + M (fx)^\rho \right]$$

as $U^\rho = d(\eta) h(\eta)$, where $d(\eta) \equiv \theta (\sigma - 1)^\rho M (fx)^\rho$ falls faster than $h(\eta) \equiv \frac{M_m}{M} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^\rho + 1$ rises, if $\eta > \frac{\beta-\rho}{\beta}$, and vice versa if $\eta < \frac{\beta-\rho}{\beta}$. In other words, the utility is maximal at $\eta = \frac{\beta-\rho}{\beta}$!

First, note that x rises as η rises, so we can rewrite both functions and look at their behavior as functions of x . Then we can compare the elasticities of these two functions and show that $\varepsilon_d < 0 < \varepsilon_h$. Moreover, if $\eta > \frac{\beta-\rho}{\beta}$ ($\eta < \frac{\beta-\rho}{\beta}$), then $|\varepsilon_d| > |\varepsilon_h|$ ($|\varepsilon_d| < |\varepsilon_h|$), so that $d(x)$ falls faster (slower) than $h(x)$ rises, and $U = d(x) h(x)$ falls (rises) as a result.

First, since $\rho < 1 < \beta$, $d(x)$ is decreasing in x :

$$d(x) = \theta (\sigma - 1)^\rho M (fx)^\rho = \theta (\sigma - 1)^\rho f^\rho \left[\frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} \right] (x)^{\rho-\beta}, \quad \text{and } \varepsilon_d = \rho - \beta < 0. \quad (46)$$

Second, $h(x) = \frac{M_m}{M} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^\rho + 1$. Note that from the (TB) condition,

$$h(x) = w m_x \frac{f_{\text{exp}}}{F_{\text{exp}}} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^\rho + 1.$$

Thus, $h(x) = 1 + \kappa(x)$, where from (42),

$$\kappa(x) = w m_x \frac{f_{\text{exp}}}{F_{\text{exp}}} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^\rho \propto y^\rho \left(\frac{x}{y} \right)^\beta \left(\frac{z}{x} \right)^\rho.$$

Moreover, since $z^{-1} \propto w^{\frac{1}{\beta}} y^{-1} = y^{\frac{\rho}{\beta}-1}$, then $\kappa(x) \propto \left(\frac{x}{y} \right)^{\beta-\rho} \left(y^{1-\frac{\rho}{\beta}} \right)^\rho = x^{\beta-\rho} y^{(\beta-\rho)\left(\frac{\rho}{\beta}-1\right)}$. Since

$\beta > 1 > \rho$ and y falls as x rises, then $\kappa(x)$, and in turn $h(x)$, is increasing with x .

$$\varepsilon_h = \frac{h'(x)}{h(x)}x = \frac{\kappa'(x)}{1 + \kappa(x)}x = \frac{\kappa'(x)}{\kappa(x)}x \frac{\kappa(x)}{1 + \kappa(x)} = \varepsilon_\kappa \frac{\kappa(x)}{1 + \kappa(x)}.$$

To calculate ε_κ , we use two properties: $\varepsilon_{a(x)b(x)} = \varepsilon_{a(x)} + \varepsilon_{b(x)}$ and $\varepsilon_{a(b(x))} = \varepsilon_{a(b)}\varepsilon_{b(x)}$. Then,

$$\varepsilon_{\kappa(x)} = \varepsilon_{x^{\beta-\rho}} + \varepsilon_{y^{(\beta-\rho)\left(\frac{\rho}{\beta}-1\right)}} = (\beta - \rho) + (\beta - \rho) \left(\frac{\rho}{\beta} - 1 \right) \varepsilon_{y(x)}.$$

From the (FE) condition, $\varepsilon_{y(x)} = -\frac{f}{f_{\text{exp}}}\left(\frac{y}{x}\right)^\beta$, so that $\varepsilon_{\kappa(x)} = (\beta - \rho) \left(1 + \left(1 - \frac{\rho}{\beta} \right) \frac{f}{f_{\text{exp}}}\left(\frac{y}{x}\right)^\beta \right)$,

$$\varepsilon_h = (\beta - \rho) \left(1 + \left(1 - \frac{\rho}{\beta} \right) \frac{f}{f_{\text{exp}}}\left(\frac{y}{x}\right)^\beta \right) \frac{\kappa(x)}{\gamma + \kappa(x)} > 0. \quad (47)$$

Finally, we can compare the absolute values of elasticities from (46) and (47):

$$\begin{aligned} |\varepsilon_d| = \beta - \rho \text{ versus } |\varepsilon_h| = (\beta - \rho) \left(1 + \left(1 - \frac{\rho}{\beta} \right) \frac{f}{f_{\text{exp}}}\left(\frac{y}{x}\right)^\beta \right) \frac{\kappa(x)}{1 + \kappa(x)}, \text{ or} \\ 1 \text{ versus } \left(1 + \left(1 - \frac{\rho}{\beta} \right) \frac{f}{f_{\text{exp}}}\left(\frac{y}{x}\right)^\beta \right) \frac{\kappa(x)}{1 + \kappa(x)}, \text{ or } \frac{1}{\kappa(x)} \text{ versus } \left(1 - \frac{\rho}{\beta} \right) \frac{f}{f_{\text{exp}}}\left(\frac{y}{x}\right)^\beta, \text{ or} \\ \frac{\beta}{\beta - \rho} \frac{f_{\text{exp}}}{f} \left(\frac{x}{y}\right)^\beta \frac{1}{\kappa(x)} \text{ versus } 1. \end{aligned} \quad (48)$$

To compare the left-hand side with 1, we plug the expressions for $\kappa(x)$ and use (43):

$$\begin{aligned} \frac{\beta}{\beta - \rho} \frac{\frac{f_{\text{exp}}}{f} \left(\frac{x}{y}\right)^\beta}{w m_x \frac{f_{\text{exp}}}{F_{\text{exp}}} \left(\frac{F_{\text{exp}} z}{f \gamma x}\right)^\rho} &= \frac{\beta}{\beta - \rho} \frac{\frac{F_{\text{exp}}}{f}}{w \left(\frac{F_{\text{exp}} z}{f \gamma x}\right)^\rho} = \frac{\beta}{\beta - \rho} \left(\frac{F_{\text{exp}}}{f}\right)^{\frac{1}{\sigma}} \left(\frac{\gamma x}{z}\right)^\rho \frac{1}{w} \\ &= \frac{\beta}{\beta - \rho} \left(\frac{F_{\text{exp}}}{f}\right)^{\frac{1}{\sigma}} \left(\frac{\gamma x}{z}\right)^\rho \frac{1}{\frac{1}{\eta} \left(\frac{F_{\text{exp}}}{f}\right)^{\frac{1}{\sigma}} \left(\frac{\gamma x}{z}\right)^\rho} = \frac{\beta}{\beta - \rho} \eta. \end{aligned}$$

Thus, the comparison in (48) results in comparing η with $\frac{\beta-\rho}{\beta}$, and we proved our results. ■

First Best Allocation and Consumption Subsidy. As shown before, the market equilibrium with a consumption subsidy $\eta = \frac{\beta-\rho}{\beta}$ satisfies:

$$\begin{aligned} (M1) : R P^{\sigma-1} \eta^{-\sigma} w^{-\sigma} (\rho x)^{\sigma-1} &= \sigma f, & (M2) : A w^{-\sigma} (\rho y)^{\sigma-1} &= \sigma f_{\text{exp}}, \\ (M3) : \int_z^\infty r_m(\varphi) M_m \mu(\varphi) d\varphi &= M_m \theta \left[\frac{F_{\text{exp}}}{\gamma} (\sigma - 1) \right], \end{aligned}$$

$$(M4) : \frac{\delta f_e}{1 - G(x)} = (f + m_{\text{exp}} f_{\text{exp}}) (\theta - 1), \quad (M5) : M\sigma\theta(f + m_{\text{exp}} f_{\text{exp}}) = 1,$$

where $(P/w)^{1-\sigma} = \theta M_m \left(\frac{\rho z}{w\gamma}\right)^{\sigma-1} + \theta M \left(\frac{\rho x}{\eta}\right)^{\sigma-1}$, i.e., we have 5 equations with 5 unknown variables, R, w, x, y, M . If we have a market equilibrium and an optimal allocation $(z_0, x_0, y_0, \alpha, \phi, M_0, \lambda, \zeta)$, which satisfies the system of equations (28)-(35), is it the case that $(x_0, y_0, z_0, M_0) = (x_M, y_M, z_M, M_M)$? The first indication that this is the case is that equations (M) and (FE) in the optimal allocation coincide with equations (M4) and (M5) in the market equilibrium.

One way to complete the answer (assuming the solutions are unique) is to postulate $(x_0, y_0, z_0, M_0) = (x_M, y_M, z_M, M_M)$ and then see if there exist $(\alpha, \phi, \lambda, \zeta)$ such that these together with (x_M, y_M, z_M, M_M) satisfy 8 equations for an optimum allocation. This is exactly the case if:

$$\phi = RP^{\sigma-1} w^{-\sigma} \rho^\sigma, \quad \alpha = Aw^{-\sigma} \rho^\sigma, \quad \zeta = \frac{\beta - \rho}{\beta} R^{\rho-1} P^{-\rho}, \quad \lambda = R^{\rho-1} P^{-\rho} \frac{\beta - \rho}{\beta} w, \quad \text{so that}$$

$$(z) : \quad \rho \frac{\beta - \rho}{\beta} z^\rho \left(\frac{(\sigma - 1) F_{\text{exp}}}{\gamma} \right)^\rho = \frac{\beta - \rho}{\beta} R^{\rho-1} P^{-\rho} \sigma F_{\text{exp}} \quad \text{or} \quad RP^{\sigma-1} \left(\frac{\gamma}{\rho z} \right)^{1-\sigma} = \sigma F_{\text{exp}},$$

(formula for import demand in ME),

$$(x) : \quad RP^{\sigma-1} w^{-\sigma} \left(\frac{\beta - \rho}{\beta} \right)^{-\sigma} \rho^\sigma x^{\sigma-1} = \sigma \rho f \quad \text{or} \quad RP^{\sigma-1} \eta^{-\sigma} \left(\frac{\rho x}{w} \right)^{\sigma-1} = \sigma w f,$$

(zero profit condition for domestic producers in ME, M1)

$$(y) : \quad Aw^{-\sigma} \rho^\sigma y^{\sigma-1} = \sigma \rho f_{\text{exp}} \quad \text{or} \quad Aw^{1-\sigma} (\rho y)^{\sigma-1} = \sigma w f_{\text{exp}},$$

(zero profit condition for exporters in ME, M2)

$$(\phi) : \quad \rho \left[RP^{\sigma-1} w^{-\sigma} \left(\frac{\beta - \rho}{\beta} \right)^{-\sigma} \rho^\sigma \right]^{\rho-1} = R^{\rho-1} P^{-\rho} \frac{\beta - \rho}{\beta} w, \quad \text{is an identity.}$$

$$(\alpha) : \quad \frac{\beta - \rho}{\beta} R^{\rho-1} P^{-\rho} A^{1/\sigma} \rho (Aw^{-\sigma} \rho^\sigma)^{\rho-1} = R^{\rho-1} P^{-\rho} \frac{\beta - \rho}{\beta} w \quad \text{is an identity.}$$

$$(M) : \quad \frac{\delta f_e}{1 - G(x)} = (\theta - 1) (f + m_{\text{exp}} f_{\text{exp}}), \quad \text{(free entry condition in ME, M4),}$$

$$(FE) : \quad 1 = \sigma\theta M (f + m_{\text{exp}} f_{\text{exp}}) \quad \text{or} \quad M = \frac{1}{\sigma\theta (f + f_{\text{exp}} m_{\text{exp}})},$$

(the expression for the mass of active firms in ME, M5)

$$(TB) : \quad \sigma F_{\text{exp}} b^\beta z^{-\beta} = M m_{\text{exp}} a \alpha^\rho y^{\sigma-1} \quad \text{or} \quad M_m F_{\text{exp}} = w M_{\text{exp}} f_{\text{exp}}.$$

(trade balance condition in ME, M3)

6.3.3 The Export Tax

The Optimal Value of Export Tax. We want to show that the optimal value of s is $\frac{\beta-\rho}{\beta} < 1$. The market equilibrium conditions are the same as those in Section 2 with $\eta = 1$:

$$(EXP) \text{ condition} \quad A s^\sigma w^{1-\sigma} (\rho y)^{\sigma-1} = \sigma w f_{\text{exp}}, \quad (49)$$

$$(FE) \text{ condition} \quad (\theta - 1) x^{-\beta} [f + m_{\text{exp}} f_{\text{exp}}] = \frac{\delta f_e}{b^\beta}, \quad (50)$$

$$(M) \text{ condition} \quad M = \frac{1}{\sigma \theta (f + m_{\text{exp}} f_{\text{exp}})} = \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} x^{-\beta}, \quad (51)$$

$$(TB) \text{ condition} \quad M_m F_{\text{exp}} = \frac{w}{s} M_{\text{exp}} f_{\text{exp}}. \quad (52)$$

$$(z) \text{ condition} \quad 1 = \sigma \theta M \left[f_{\text{exp}} m_{\text{exp}} + \frac{F_{\text{exp}}}{w} \left(\gamma \frac{x}{wz} \right)^{\sigma-1} \right]. \quad (53)$$

Now we are ready to prove that $s = \frac{\beta-\rho}{\beta}$ **results in the maximal level of welfare.**

Proof. Step 1. First, note that when s rises, y must fall and x must rise. The proof is the same as in Section 6.3.2, with equations (42), (43), and (45) rewritten as

$$(EXP): \quad w^\sigma = \frac{A \rho^{\sigma-1}}{\sigma f_{\text{exp}}} s^\sigma y^{\sigma-1}, \quad \frac{F_{\text{exp}}}{w} \left(\frac{\gamma x}{wz} \right)^{\sigma-1} = f, \quad \text{and} \quad w^{\sigma-\frac{\sigma-1}{\beta}} \propto \left(\frac{x}{y} \right)^{\sigma-1}. \quad (54)$$

Step 2. This step is also the same as in Section 6.3.2 with $\kappa(x) = \frac{w}{s} m_x \frac{f_{\text{exp}}}{F_{\text{exp}}} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^\rho$, and when we need to compare

$$\frac{\beta}{\beta - \rho} \frac{f_{\text{exp}}}{f} \left(\frac{x}{y} \right)^\beta \frac{1}{\kappa(x)} \quad \text{versus} \quad 1,$$

from using (54) instead of (43) and the new $\kappa(x)$ function, we get:

$$\begin{aligned} \frac{\beta}{\beta - \rho} \frac{\frac{f_{\text{exp}}}{f} \left(\frac{x}{y} \right)^\beta}{\frac{w}{s} m_x \frac{f_{\text{exp}}}{F_{\text{exp}}} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^\rho} &= \frac{\beta}{\beta - \rho} \left(\frac{F_{\text{exp}}}{f} \right)^{\frac{1}{\sigma}} \left(\frac{\gamma x}{z} \right)^\rho \frac{s}{w} \\ &= \frac{\beta}{\beta - \rho} \left(\frac{F_{\text{exp}}}{f} \right)^{\frac{1}{\sigma}} \left(\frac{\gamma x}{z} \right)^\rho \frac{s}{\left(\frac{F_{\text{exp}}}{f} \right)^{\frac{1}{\sigma}} \left(\frac{\gamma x}{z} \right)^\rho} = \frac{\beta}{\beta - \rho} s. \end{aligned} \quad (55)$$

Thus, the comparison above results in comparing s with $\frac{\beta-\rho}{\beta}$, and we proved our results. ■

First Best Allocation and Export Tax As in Section 6.3.2, it can be shown that the market equilibrium conditions for $s = \frac{\beta - \rho}{\beta}$ coincide with the system of equations (28)-(35), if

$$\phi = RP^{\sigma-1} \left(\frac{\rho}{w} \right)^\sigma, \quad \alpha = A \left(\frac{\rho}{w} \right)^\sigma \left(\frac{\beta - \rho}{\beta} \right)^\sigma, \quad \lambda = R^{\rho-1} P^{-\rho} w, \quad \zeta = \frac{\beta - \rho}{\beta} R^{\rho-1} P^{-\rho}.$$

6.3.4 The Import Tariff

The Optimal Value of Import Tariff. The derivations of the equilibrium conditions with the import tariff t are very similar to those in Section 2 with $\eta = 1$ and $s = 1$. As a result, we have

$$\text{(EXP) condition} \quad Aw^{1-\sigma} (\rho y)^{\sigma-1} = \sigma w f_{\text{exp}}, \quad \text{and} \quad (56)$$

$$\text{(FE) condition} \quad (\theta - 1) x^{-\beta} [f + m_{\text{exp}} f_{\text{exp}}] = \delta f_e / b^\beta. \quad (57)$$

$$\text{(TB) condition} \quad M_m F_{\text{exp}} = w M_{\text{exp}} f_{\text{exp}}$$

We need to derive the zero profit condition for importers. The demand for the foreign variety v is $q_m(v) = RP^{\sigma-1} (tp_m(v))^{-\sigma}$. The expenditures in the economy are $R = w + T$, where T is:

$$T = M_m \int_z \frac{(t-1)}{t} [tp(\varphi)] \mu(\varphi) d\varphi = (t-1) [RP^{\sigma-1} t^{-\sigma}] \theta M_m \left(\frac{\rho z}{\gamma} \right)^{\sigma-1},$$

where $P^{1-\sigma} = \theta \left[M_m \left(\frac{\rho z}{\gamma t} \right)^{\sigma-1} + M \left(\frac{\rho x}{w} \right)^{\sigma-1} \right]$. Then $R = \frac{w}{1 - \frac{t-1}{t} P^{\sigma-1} \theta M_m \left(\frac{\rho z}{\gamma t} \right)^{\sigma-1}}$, and the new (z) condition can be written as

$$RP^{\sigma-1} \frac{1}{t} \left(\frac{\rho z}{\gamma t} \right)^{\sigma-1} = \sigma F_{\text{exp}} \iff \frac{w \frac{1}{t} \left(\frac{\rho z}{\gamma t} \right)^{\sigma-1}}{P^{1-\sigma} - \frac{t-1}{t} \theta M_m \left(\frac{\rho z}{\gamma t} \right)^{\sigma-1}} = \sigma F_{\text{exp}},$$

$$\text{or } \frac{w}{\sigma \theta F_{\text{exp}}} = M_m + M t^\sigma \left(\frac{\gamma x}{w z} \right)^{\sigma-1},$$

or using the (TB) condition,

$$\text{(z) condition} \quad \frac{w}{\sigma \theta F_{\text{exp}}} = M_x \frac{w f_{\text{exp}}}{F_{\text{exp}}} + M t^\sigma \left(\frac{\gamma x}{w z} \right)^{\sigma-1}.$$

Finally,

$$\text{(M) condition:} \quad M = \frac{1}{\sigma \theta (f + m_{\text{exp}} f_{\text{exp}})} = \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} x^{-\beta} \quad (58)$$

Now we are ready to prove that **an import tariff $t = \frac{\beta}{\beta - \rho}$ maximizes welfare.**

Proof. Step 1. First, we prove that when t rises, x falls, while y and w rise.

From the (FE) condition, x and y must move in the opposite direction. Assume that y falls.

Then x rises and from

$$\text{(EXP): } \frac{w^\sigma}{y^{\sigma-1}} = \frac{A\rho^{\sigma-1}}{\sigma f_{\text{exp}}}, \quad (59)$$

w must fall. On the other hand, from

$$\begin{aligned} \text{(z) + (M)} \quad \Rightarrow \quad \frac{w}{F_{\text{exp}}} &= \frac{1}{f + m_{\text{exp}}f_{\text{exp}}} \left[m_{\text{exp}} \frac{w f_{\text{exp}}}{F_{\text{exp}}} + t^\sigma \left(\frac{\gamma x}{wz} \right)^{\sigma-1} \right], \quad \text{or} \\ \frac{w}{F_{\text{exp}}} &= \frac{w}{F_{\text{exp}}} + \frac{1}{f + m_{\text{exp}}f_{\text{exp}}} \left[t^\sigma \left(\frac{\gamma x}{wz} \right)^{\sigma-1} - \frac{w f}{F_{\text{exp}}} \right], \quad \text{or} \\ t^\sigma \left(\frac{\gamma x}{wz} \right)^{\sigma-1} &= \frac{w f}{F_{\text{exp}}} \quad \text{or} \quad w^\sigma = t^\sigma \frac{F_{\text{exp}}}{f} \left(\frac{\gamma x}{z} \right)^{\sigma-1}. \end{aligned} \quad (60)$$

Again, from (TB) condition, $z^{-1} = \left(\frac{f_{\text{exp}}}{F_{\text{exp}}} w M_{\text{exp}} \right)^{\frac{1}{\beta}} \propto w^{\frac{1}{\beta}} y^{-1}$ so that

$$w \propto \left[t^\sigma \left(\frac{x}{y} \right)^{\sigma-1} \right]^{\frac{1}{\sigma - \frac{1}{\beta}}}, \quad (61)$$

and from (61), w rises, which contradicts to the previous conclusion about w . Thus, we proved that y cannot fall with an increase in t , and as t rises, y and w rise as well, and x falls.

Step 2. Now we are ready to derive the optimal import tariff. Note that

$$U^\rho = \int_z^\infty [q_m(\varphi)]^\rho M_m \mu(\varphi) d\varphi + \int_x^\infty q(\varphi)^\rho M \mu(\varphi) d\varphi = \theta (\sigma - 1)^\rho \left[M_m \left(\frac{F_{\text{exp}} z}{\gamma} \right)^\rho + M (fx)^\rho \right]$$

We will show that $U^\rho = d(x) h(x)$, where as x rises, $d(x) \equiv \theta (\sigma - 1)^\rho M (fx)^\rho$ falls faster than $h(x) \equiv \frac{M_m}{M} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^\rho + 1$ rises, if $t < \frac{\beta}{\beta - \rho}$, and the opposite happens, if $t > \frac{\beta}{\beta - \rho}$. In other words, $U(x)$ falls with x , if $t < \frac{\beta}{\beta - \rho}$, and it rises with x , if $t > \frac{\beta}{\beta - \rho}$. Then since $dx/dt < 0$,

$$\frac{dU}{dt} = \frac{dU}{dx} \frac{dx}{dt} = \begin{cases} > 0, & \text{if } t < \frac{\beta}{\beta - \rho}, \\ < 0, & \text{if } t > \frac{\beta}{\beta - \rho}, \end{cases}$$

and the utility reaches its maximum, when $t = \frac{\beta}{\beta - \rho}$.

We can compare the elasticities of $d(x)$ and $h(x)$ and show that $\varepsilon_d > 0 > \varepsilon_h$. Thus, the behavior of $U^\rho = d(x) h(x)$ depends on the comparison of absolute terms $|\varepsilon_d|$ and $|\varepsilon_h|$. First, note that $\varepsilon_{d(x)} = \rho - \beta < 0$. In addition, from the (TB) condition,

$$h(x) = w m_x \frac{f_{\text{exp}}}{F_{\text{exp}}} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^\rho + 1 = \kappa(x) + 1,$$

where from $w \propto y^{\frac{1}{\rho}}$ and $z^{-1} \propto w^{\frac{1}{\beta}} y^{-1}$,

$$\kappa(x) = wm_x \frac{f_{\text{exp}}}{F_{\text{exp}}} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^{\rho} \propto \left(\frac{x}{y} \right)^{\beta - \rho} \left(y^{1 - \frac{\rho}{\beta}} \right)^{\rho} = x^{\beta - \rho} y^{(\beta - \rho) \left(\frac{\rho}{\beta} - 1 \right)}, \text{ and}$$

$$\varepsilon_{h(x)} = \frac{h'(x)}{h(x)} x = \frac{\kappa'(x)}{1 + \kappa(x)} x = \frac{\kappa'(x)}{\kappa(x)} x \frac{\kappa(x)}{1 + \kappa(x)} = \varepsilon_{\kappa(x)} \frac{\kappa(x)}{1 + \kappa(x)},$$

$$\text{where } \varepsilon_{\kappa(x)} = (\beta - \rho) + (\beta - \rho) \left(\frac{\rho}{\beta} - 1 \right) \varepsilon_{y(x)}, \text{ and}$$

$$\varepsilon_h = (\beta - \rho) \left(1 + \left(1 - \frac{\rho}{\beta} \right) \frac{f}{f_{\text{exp}}} \left(\frac{y}{x} \right)^{\beta} \right) \frac{\kappa(x)}{\gamma + \kappa(x)} > 0.$$

Finally, we can compare the absolute values of elasticities:

$$|\varepsilon_d| = \beta - \rho \text{ versus } |\varepsilon_h| = (\beta - \rho) \left(1 + \left(1 - \frac{\rho}{\beta} \right) \frac{f}{f_{\text{exp}}} \left(\frac{y}{x} \right)^{\beta} \right) \frac{\kappa(x)}{1 + \kappa(x)}, \text{ or}$$

$$\frac{\beta}{\beta - \rho} \frac{f_{\text{exp}}}{f} \left(\frac{x}{y} \right)^{\beta} \frac{1}{\kappa(x)} \text{ versus } 1.$$

To compare the left-hand side with 1, we plug the expressions for $\kappa(x)$ and use $w^{\sigma} = t^{\sigma} \frac{F_{\text{exp}}}{f} \left(\frac{\gamma x}{z} \right)^{\sigma - 1}$:

$$\frac{\beta}{\beta - \rho} \frac{\frac{f_{\text{exp}}}{f} \left(\frac{x}{y} \right)^{\beta}}{wm_x \frac{f_{\text{exp}}}{F_{\text{exp}}} \left(\frac{F_{\text{exp}} z}{f \gamma x} \right)^{\rho}} = \frac{\beta}{\beta - \rho} \left(\frac{F_{\text{exp}}}{f} \right)^{\frac{1}{\sigma}} \left(\frac{\gamma x}{z} \right)^{\rho} \frac{1}{w} = \frac{\beta}{\beta - \rho} \frac{1}{t}. \quad (62)$$

Thus, we proved our results. ■

First Best Allocation and Import Tariff As in Section 6.3.2 it can be shown that the market equilibrium conditions for $t = \frac{\beta}{\beta - \rho}$ coincide with system of equations (28)-(35), if

$$\phi = RP^{\sigma - 1} \left(\frac{\rho}{w} \right)^{\sigma}, \quad \alpha = A \left(\frac{\rho}{w} \right)^{\sigma}, \quad \lambda = R^{\rho - 1} P^{-\rho} w, \quad \zeta = R^{\rho - 1} P^{-\rho}.$$

6.4 Proof of Propositions 2 and 3

The market equilibrium conditions for the case of the export subsidy in the presence of the consumption subsidy are derived in Section 3. Hereafter, we assume that the government has in place the optimal consumption subsidy (i.e., $\eta = \frac{\beta - \rho}{\beta}$) and explore how export subsidies affect the three components of the utility function by first proving Proposition 2 and then proving Proposition 3.

Proof. The proofs for Propositions 2 and 3 are the same as in Section 6.3.3 with the comparison of $\varepsilon_{h(x)}$ and $\varepsilon_{d(x)}$ resulting in the comparison $s \geq 1$. ■

6.5 Quantitative Exercise for Two Components of Utility Function

We want to study the behavior of TOT and variety indices. We show below that anything is possible, i.e., there are 3 cases: (Which case happens depends on the parameters.)

(1) TOT index falls, Variety index falls;

This happens if we set the parameters:

$$\begin{aligned}\delta &= 0.01, \quad \beta = 4, \quad \sigma = 3.8, \quad b = 1, \\ \frac{f_{\text{exp}}}{f} &= 15000, \quad \frac{F_{\text{exp}}}{f} = 1500, \quad \frac{f_e}{f} = 30, \quad \frac{A}{f} = 200000, \quad \frac{L}{f} = 0.5,\end{aligned}$$

and vary s between 0.3 and 2.7. Then $\beta > \sigma$, $y > x > b$, $M_e > 0$, $wage > 0$, and both indices fall.

(2) TOT index falls, Variety index rises;

Compared to case (1), the only parameters needed to be changed to get such behavior of indices are $\frac{f_{\text{exp}}}{f} = 150000$ and $\frac{F_{\text{exp}}}{f} = 1.5$.

(3) TOT index rises, Variety index falls;

Again, compared to case (1), we change only 2 parameters: now $\frac{f_{\text{exp}}}{f} = 15$ and $\frac{F_{\text{exp}}}{f} = 15000$.