Estimation of Multiple Output Production Functions

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Abstract

This paper addresses specification and estimation of multiple-outputs and multiple-inputs production technology in the presence of technical inefficiency. Our primary focus is on the primal formulations. Several competing specifications such as production function, input (output) distance function, input requirement function are considered. We show that all these specifications come from the same transformation function and are algebraically identical. We also show that:(i) unless the transformation function is separable (i.e., outputs are separable from inputs), the input (output) ratios in the input (output) distance function can not be treated as exogenous (uncorrelated with technical inefficiency) resulting inconsistent estimates of the input (output) distance function parameters. (ii) Even if input (output) ratios are exogenous, estimation of the input (output) distance function will result in inconsistent parameter estimates if outputs (inputs) are endogenous. (iii) In the translog model with production uncertainty, input (output) ratios are not exogenous even if outputs (inputs) are exogenous which will make the distance function parameters inconsistent. We address endogeneity and instrumental variable issues in details in the context of both restricted (Cobb-Douglas and Constant-Elasticity of Transformation) and flexible (translog) functional forms. Estimation of several specifications using both single and system approaches are discussed using Norwegian dairy farming data.

Keywords: Production function, distance function, input requirement function, Transformation function, Cobb-Douglas and Translog functions.

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1 Introduction

Although most of the production processes involve multiple inputs and multiple outputs, estimation of multiple inputs and multiple outputs production function is not popular especially when a single equation approach is used. The problem is that in estimating a production function (using OLS or nonlinear least squares, NLS) one of the output (usually the dependent variable) is considered endogenous[1] and the rest of them (along with the inputs) are treated as exogenous. If all outputs are endogenous, estimation results using the production function suffer from the endogeneity problem since endogeneity of only one output is recognized, i.e., the one used as the dependent variable. Furthermore, results differ depending on which output is chosen as the dependent variable. If the inputs are also endogenous, the endogeneity problem is magnified because in this case all the regressors will be endogenous. To avoid this problem researchers use distance function formulations while estimating multiple-inputs and multiple-output technologies. However, the distance functions cannot avoid the endogeneity problem completely. For example, in estimating the input distance function (IDF) (Shephard (1953, 1970)) the maintained hypothesis (standard assumption) is that outputs are exogenously given. While this might be true for service and demand determined industries such as banks, airlines, railroads, post offices and public utilities, outputs for manufacturing firms and agricultural farms are unlikely to be exogenously given (except perhaps the case when there are explicit quotas on outputs). Thus, results from the IDF models might also suffer from endogeneity problem, especially if outputs are endogenous. Similarly, in estimating output distance functions (ODF) the implicit assumption is that inputs are exogenously given (which is rarely the case in practice). Thus the ODF results will be inconsistent if inputs are not exogenous. Since the outputs (inputs) are assumed to be exogenous (endogenous) in IDF (ODF), one needs to check appropriateness of the endogeneity/exogeneity assumptions either from econometric or from economic considerations. The question then is: if outputs (inputs) are endogenous, can one treat the input (output) ratios as exogenous in the IDF (ODF)? If so one needs instruments for the output (input) variables in the IDF (ODF) models. If not, instruments for all input and output variables are required irrespective of whether an IDF or an ODF is used. Although there are papers on input and output distance functions in which instrumental variables are used, it is not clear which regressors are endogenous and in what specifications. That is, the readers do not get a clear idea of whether to worry about endogeneity of input ratios or outputs or both while using an IDF.

In this paper we address the endogeneity issue from the behavioral assumption that producers maximize

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[1] An input (output) is said to be endogenous if it is correlated with the error term in the estimating equation. Since we model inefficiency in production explicitly via a random variable in addition to the usual random noise, an input (output) can be endogenous if it is uncorrelated with the noise component but correlated with the inefficiency component.

profit and both inputs and outputs are decision/choice variables. We focus mostly on the distance function formulations because the endogeneity problem is obvious for the production and input requirement functions (in which inputs and outputs appear as regressors) but is not so obvious for distance functions since some of the regressors appear in ratio form.[2] Färe and Primont (1995) dealt with the theoretical issues associated with multi-output distance function models in details. However, they did not discuss estimation issues and therefore the endogeneity issue never arose in their discussion. Kumbhakar and Lovell (2000) discussed the issues in terms of the dual profit function. Our primary concern here is estimation and we examine the endogeneity issue in details using both restricted and flexible forms of the transformation function (instead of a dual profit function) from which the IDF and ODF are derived. The idea is check when and where one can use the input and output ratios as exogenous. First, we start with a separable and restrictive representation of the transformation function (Cobb-Douglas (CD) input and constant elasticity of transformation (CET) output functions) and show that even in this case endogeneity of regressors appear in both the IDF and ODF models. We then consider the translog formulation to show that the endogeneity problem stays even the technology exhibits constant returns to scale (CRTS). Under CRTS the endogeneity problem can be avoided if the IDF (ODF) is rewritten where the regressors are in ratio form. However, these formulations are not the standard IDF (ODF) models that are used in the literature. Finally, we consider production uncertainty and show that under expected profit maximization behavior output (input) ratios are not even exogenous in the ODF (IDF) when inputs (outputs) are exogenous.

The rest of the paper is organized as follows. In Section 2 we deal with specification of different formulations starting from a restrictive transformation function, which is then generalized to accommodate flexible functional form. Estimation issues of these models are discussed in Section 3 from a single equation perspective. Section 4 addresses estimation issues using a system approach. Section 5 discusses estimation issues with production uncertainty. The data used in the paper is discussed in Section 6 and Section 7 reports empirical results. Finally, Section 8 summarizes the main results and conclusions of the paper.

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[2] Kumbhakar (1996) addressed the endogeneity issue in a multiple production model with technical and allocative inefficiency in terms of a profit function. The main problems in dealing with translog profit function are the following. First, it cannot handle negative profit which is quite common in reality; second, estimation of profit function relies exclusively on input and output prices which are often difficult to get and variations in prices are often very little (which makes the parameter estimates imprecise). Because of these problems we formulate the problem in a primal framework so that information on input and output quantities can be directly used.

2 Representations of the transformation function

2.1 The Cobb-Douglas case

Consider a production process in which M outputs are produced using J inputs and the technology is specified as $A\, f(\theta x, \lambda y) = 1$, where x is a vector of K inputs and y is a vector of M outputs. The A term captures the impact of observed and unobserved factors that affect the transformation function neutrally. Input technical inefficiency is indicated by $\theta \leq 1$ and output technical inefficiency is captured by $\lambda \geq 1$ (both are scalars). Thus, $\theta x \leq x$ is the input vector in efficiency (effective) units so that if $\theta = 0.9$ inputs are 90% efficient (i.e., use of each input could be reduced by 10% without reducing outputs, if inefficiency is eliminated). Similarly, if λ is 1.05, each output could be increased by 5% without increasing any input, when inefficiency is eliminated. Since both θ and λ are not identified, we consider the following special cases. If $\theta = 1$ and $\lambda > 1$ then we have output-oriented technical inefficiency. Similarly, if $\lambda = 1$ and $\theta < 1$ then we have input-oriented technical inefficiency. Finally, if $\lambda \cdot \theta = 1$ technical inefficiency is said to be hyperbolic. It says that if the inputs are contracted by a constant proportion outputs are expanded by the same proportion. That is, instead of moving to the frontier by either expanding outputs (keeping the inputs unchanged) or contracting inputs (holding outputs unchanged), the hyperbolic measure chooses a path to the frontier that leads to a simultaneous increase in outputs and a decrease in inputs by the same rate. We specify the technology in terms of the transformation function $f(\cdot)$ because it is much more general than the production/distance/input requirement function.

To fix ideas we start from the case where the transformation function is separable (i.e., the output function is separable from the input function) so that $A\, f(\theta x, \lambda y) = 1$ can be rewritten as $A\, g(\lambda y) \cdot h(\theta x) = 1$. Furthermore, if we assume that both $g(\cdot)$ and $h(\cdot)$ are Cobb-Douglas (to be relaxed later), the transformation function can be expressed as

$$\textbf{CD transformation functions:}\ A \prod_m \{\lambda y_m\}^{\alpha_m} \prod_j \{\theta x_j\}^{\beta_j} = 1. \tag{1}$$

If we normalize $\alpha_1 = -1, \theta = 1$ then we get the following specification, viz.,

$$\textbf{Production function:}\ y_1 = A \prod_{m=2} y_m^{\alpha_m} \prod_j x_j{}^{\beta_j}\ \lambda^{\sum_m \alpha_m} \tag{2}$$

which can be viewed as a production function. Output-oriented technical efficiency in this model is $TE = \lambda^{\sum_m \alpha_m}$ and output-oriented technical inefficiency is $u = \ln TE = \{\sum_m \alpha_m\} \ln \lambda < 0$ since in (2) $\ln \lambda > 0$

and $\alpha_m < 0 \; \forall \; m \Rightarrow \sum \alpha_m < 0$.

If we rewrite (1) as

$$A \, y_1^{\sum_m \alpha_m} \prod_{m=2} \{y_m/y_1\}^{\alpha_m} \prod_j x_j^{\beta_j} \; \theta^{\sum_j \beta_j} \; \lambda^{\sum_m \alpha_m} = 1 \tag{3}$$

and use the normalization $\sum_m \alpha_m = -1, \theta = 1$, then we get the output distance function (ODF) formulation (Shephard (1953)), viz.,

$$\textbf{Output distance function:} \;\; y_1 = A \prod_{m=2} \{y_m/y_1\}^{\alpha_m} \prod_j x_j^{\beta_j} \; \lambda^{-1} \tag{4}$$

where output-oriented technical inefficiency $u = -\ln \lambda < 0$. Technical inefficiency in these models (2) and (4) are different because the output variables (as regressors) appear differently and different normalizations are used.

Similarly we rewrite (1) as

$$A \, x_1^{\sum_j \beta_j} \prod_m y_m^{\alpha_m} \prod_{j=2} \{x_j/x_1\}^{\beta_j} \; \theta^{\sum_j \beta_j} \; \lambda^{\sum_m \alpha_m} = 1 \tag{5}$$

and use the normalization $\sum_j \beta_j = -1, \lambda = 1$, to get the input distance function (IDF) formulation (Shephard (1953)), viz.,

$$\textbf{Input distance function:} \;\; x_1 = A \prod_m y_m^{\alpha_m} \prod_{j=2} \{x_j/x_1\}^{\beta_j} \; \theta^{-1} \tag{6}$$

where input-oriented technical inefficiency is $u = -\ln \theta > 0$ which is the percentage over-use of inputs due to inefficiency.

To get the hyperbolic measure of inefficiency from the above IDF all we need to do is to use the normalization $\sum_j \beta_j = -1$ and $\lambda = \theta^{-1}$ in (5) which gives the **hyperbolic input distance function** (Färe et al. (1995), Cuesta and Zofio (2005))

$$x_1 = A \prod_m y_m^{\alpha_m} \prod_{j=2} \{x_j/x_1\}^{\beta_j} \; \lambda^{\{1+\sum_m \alpha_m\}} \tag{7}$$

Since (6) and (7) are identical algebraically, $-\ln \theta$ in (6) is the same as $(1 + \sum_m \alpha_m) \ln \lambda$ in (7), and one can get $\ln \lambda$ after estimating inefficiency from either of these two equations.

Finally, if we use the normalization $\beta_1 = -1, \lambda = 1$ in (1) it can be written as

$$\textbf{Input Requirement function:} \;\; x_1 = A \prod_m y_m^{\alpha_m} \prod_{j=2} x_j^{\beta_j} \; \theta^{\sum_j \beta_j} \tag{8}$$

which is the input requirement function (IRF) due to Diewert (1974). Input-oriented technical inefficiency in this model is $u = \{\sum_j \beta_j\} \ln \theta > 0$ since $\beta_j < 0$ and $\ln \theta < 0$.

Note that all these specifications are algebraically the same in the sense that if the technology is known inefficiency can be computed from any one of these specifications. However, empirical results are not invariant to the specification used.

2.2 The CET-CD technology

Since the CD output function does not satisfy the second order (concavity) condition for profit maximization, we replace the CD output function by the constant elasticity of transformation (CET) output function (Powell and Gruen (1968)), viz., $g(\lambda y) = [\sum \delta_m (\lambda y_m)^c]^{1/c}$, $\delta_m \geq 0, \sum_m \delta_m = 1, c > 1$. For the CET function the elasticity of transformation between any two outputs is $1/(1-c)$. With this specification the transformation function can be expressed as

$$\alpha_0 + (1/c) \ln \{ \sum_{m=1} \delta_m y_m^c \} + \sum_j \beta_j \ln x_j + u = 0, \ \beta_j < 0, \tag{9}$$

where $u = \ln \lambda + (\sum_j \beta_j) \ln \theta$.

If we rewrite (9) as

$$-\ln y_1 = \alpha_0 + (1/c) \ln \left[\delta_1 + \sum_{m=2} \delta_m (y_m/y_1)^c \right] + \sum_j \beta_j \ln x_j + u \tag{10}$$

then it can be viewed as an output distance function.[3] If we normalize $\theta = 1$ then the inefficiency terms becomes $u = \ln \lambda$.

On the other hand, If we rewrite (9) as

$$-\ln x_1 = \{1/\sum_j \beta_j\}\{\alpha_0 + (1/c) \ln \left[\sum_{m=1} \delta_m y_m^c \right] + \sum_{j=2} \beta_j \ln(x_j/x_1)\} + u\{1/\sum_j \beta_j\} \tag{11}$$

and normalize $\lambda = 1$ then it can be viewed as an input distance function in which the inefficiency term becomes $u = \ln \theta$.

Finally, if we normalize $\ln \lambda = -\ln \theta$ in (11) we get the hyperbolic input distance function formulation, viz.,

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[3] Note that this can also be viewed as a production function. The functional form of $g(\cdot)$ is such that when one output is taken out of $g(\cdot)$ the other outputs are automatically expressed in ratio form. Thus the ODF cannot be separated from the production function.

$$-\ln x_1 = \{1/\sum_j \beta_j\}\{\alpha_0 + (1/c)\ln\left[\sum_{m=1}\delta_m y_m^c\right] + \sum_{j=2}\beta_j \ln(x_j/x_1)\} + u_h \qquad (12)$$

where $u_h = \ln\lambda\{1-(\sum_j \beta_j)\}/\{\sum_j \beta_j\}$. On the other hand, using the normalization $\ln\lambda = -\ln\theta$ in (10) we get the hyperbolic output distance function formulation, viz.,

$$-\ln y_1 = \alpha_0 + (1/c)\ln\left[\delta_1 + \sum_{m=2}\delta_m (y_m/y_1)^c\right] + \sum_j \beta_j \ln x_j + u_{ho} \qquad (13)$$

where $u_{ho} = \ln\lambda(1-\sum_j \beta_j)$. In the hyperbolic model we want to estimate either $\ln\lambda$ or $\ln\theta$. Thus if either u_h or u_{ho} is estimated, one can get $\ln\lambda$ or $\ln\theta$.

We can also rewrite (9) as

$$\ln x_1 = -\{1/\beta_1\}\{\alpha_0 + (1/c)\ln\{\sum \delta_m y_m^c\} + \sum_j \beta_j \ln x_j\} + u \qquad (14)$$

where $u = -\{1/\beta_1\}(\ln\lambda + (\sum_j \beta_j)\ln\theta) = -\ln\theta\{\sum_j \beta_j\}/\beta_1$ after normalizing $\lambda = 1$. This can be viewed as the IRF.

Note that all the above formulations are derived from (9) and are therefore algebraically the same. Therefore, it is not necessary to estimate all of these models.[4] Estimated inefficiency from any of the above models can be used to obtain inefficiency in other models. That is, if one estimates output-oriented inefficiency from (10), it can be easily converted to input-oriented and hyperbolic inefficiency. More specifically, u in (10) is $\ln\lambda > 0$ when it is viewed as output-oriented measure. We get the input-oriented measure $-\ln\theta > 0$ from $-u/\sum_j \beta_j$ and the hyperbolic measure $\ln\lambda = -\ln\theta$ from $u/(1-\sum_j \beta_j)$. Note that in the CET-CD case, RTS is $-\sum_j \beta_j$ so that all the above inefficiency measures are positive. Thus, the link among different inefficiency measures are made via RTS (which is constant in the CET-CD case).

To show whether the above results hold for more flexible functional forms as well, we now consider the translog functional form for the transformation function.

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[4] In some of the specifications only a subset of regressors are endogenous while in others all the regressors are endogenous. We discuss estimation issues later.

2.3 The translog transformation function

We write the transformation function as $A\, f(y^*, x^*) = 1$, where $y^* = y\lambda$, $x^* = x\theta$, and $f(y^*, x^*)$ is assumed to be translog (TL), i.e.,

$$\textbf{TL transformation function:}\ \ \ln f(y^*, x^*) = \sum_m \alpha_m \ln y_m^* + \frac{1}{2}\sum_m \sum_n \alpha_{mn} \ln y_m^* \ln y_n^*$$
$$+ \sum_j \beta_j \ln x_j^* + \frac{1}{2}\sum_j \sum_k \beta_{jk} \ln x_j^* \ln x_k^* + \sum_m \sum_j \delta_{mj} \ln y_m^* \ln x_j^*, \tag{15}$$

The above function is assumed to satisfy the following symmetry restrictions, viz., $\beta_{jk} = \beta_{kj}$ and $\alpha_{mn} = \alpha_{nm}$. Although it does not make much sense to define a production function (one output as the dependant variable), one can use the following normalizations $(\alpha_1 = -1, \alpha_{1n} = 0, \forall\, n, \delta_{1j} = 0, \forall\, j, \theta = 1)$ to obtain a pseudo production function, viz.,

$$\textbf{TL Production function:}\ \ \ln y_1 = \alpha_0 + \sum_j \beta_j \ln x_j + \frac{1}{2}\sum_j \sum_k \beta_{jk} \ln x_j \ln x_k$$
$$+ \sum_{m=2} \alpha_m \ln y_m + \frac{1}{2}\sum_{m=2} \sum_{n=2} \alpha_{mn} \ln y_m \ln y_n + \sum_{m=2} \sum_j \delta_{mj} \ln y_m \ln x_j + u \tag{16}$$

where $u = \ln\lambda(-1 + \sum_{m=2}\alpha_m + \sum_{m=2}\sum_{n=2}\alpha_{mn}\ln y_n + \sum_{m=2}\sum_j \delta_{mj}\ln x_j) + \frac{1}{2}\sum_{m=2}\sum_{n=2}\alpha_{mn}(\ln\lambda)^2$.

If we rewrite (15) as

$$\ln f(y^*, x^*) = \sum_{m=2} \alpha_m \ln(y_m/y_1) + \frac{1}{2}\sum_{m=2}\sum_{n=2} \alpha_{mn} \ln(y_m/y_1)\ln(y_n/y_1) + \sum_j \beta_j \ln x_j^* + \frac{1}{2}\sum_j \sum_k \beta_{jk} \ln x_j^* \ln x_k^*$$
$$+ \sum_{m=2}\sum_j \delta_{mj} \ln x_j^* \ln(y_m/y_1) + \big[\sum_m \alpha_m\big] \ln y_1^* + \sum_m \big[\sum_n \alpha_{mn}\big] \ln y_m \ln y_1^* + \sum_j \big[\sum_m \delta_{mj}\big] \ln x_j^* \ln y_1^*$$

and use the following normalizations $\sum_m \alpha_m = -1, \sum_n \alpha_{mn} = 0, \forall\, m, \sum_m \delta_{mj} = 0\,, \forall\, j, \theta = 1$, we obtain the output distance function representation,[5] viz.,

$$\textbf{TL ODF:}\ \ \ln y_1 = \alpha_0 + \sum_j \beta_j \ln x_j + \frac{1}{2}\sum_j \sum_k \beta_{jk} \ln x_j \ln x_k$$
$$+ \sum_{m=2} \alpha_m \ln \hat{y}_m + \frac{1}{2}\sum_{m=2}\sum_{n=2} \alpha_{mn} \ln \hat{y}_m \ln \hat{y}_n + \sum_j \sum_{m=2} \delta_{mj} \ln x_j \ln \hat{y}_m + u, \tag{17}$$

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[5] Note that these identifying/normalizing constraints make the transformation function homogeneous of degree one in outputs. In the efficiency literature one starts from a distance function (which is the transformation function with inefficiency built in) and imposes linear homogeneity (in outputs) constraints to get the ODF. Here we get the same end-result without using the notion of a distance function to start with.

where $u = -\ln\lambda < 0, \hat{y}_m = y_m/y_1, m = 2, \cdots, M$.

Furthermore, if we rewrite (15) as

$$\begin{aligned}\ln f(y^*, x^*) = &\sum_m \alpha_m \ln y_m^* + \frac{1}{2}\sum_m\sum_n \alpha_{mn} \ln y_m^* \ln y_n^* + \sum_{j=2} \beta_j \ln(x_j/x_1) \\ &+ \frac{1}{2}\sum_{j=2}\sum_{k=2} \beta_{jk} \ln(x_j/x_1)\ln(x_k/x_1) + \sum_m\sum_{j=2} \delta_{mj} \ln(x_j/x_1)\ln y_m^* + \big[\sum_j \beta_j\big] \ln x_1^* \\ &+ \sum_j \big[\sum_k \beta_{jk}\big] \ln x_j \ln x_1^* + \sum_m \big[\sum_j \delta_{mj}\big] \ln y_m^* \ln x_1^* \end{aligned} \quad (18)$$

and use the following normalizations $\sum_j \beta_j = -1, \sum_k \beta_{jk} = 0, \forall\, j, \sum_j \delta_{mj} = 0, \forall\, m, \lambda = 1$, we get the input distance function representation,[6] viz.,

$$\begin{aligned}\textbf{TL IDF:}\quad \ln x_1 = &\alpha_0 + \sum_{j=2} \beta_j \ln \hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2} \beta_{jk} \ln \hat{x}_j \ln \hat{x}_k + \sum_m \alpha_m \ln y_m \\ &+ \frac{1}{2}\sum_m\sum_n \alpha_{mn} \ln y_m \ln y_n + \sum_m\sum_{j=2} \delta_{mj} \ln \hat{x}_j \ln y_m + u,\end{aligned} \quad (19)$$

where $u = -\ln\theta > 0, \hat{x}_j = x_j/x_1, j = 2, \cdots, J$.

To get to the hyperbolic specification in the above IDF we start from (18) and use the normalization $\ln\lambda = -\ln\theta$ in addition to $\sum_j \beta_j = -1, \sum_k \beta_{jk} = 0, \forall\, j, \sum_j \delta_{mj} = 0, \forall\, m$. This gives the **hyperbolic specification** of the IDF, viz.,

$$\begin{aligned}\ln x_1 = &\alpha_0 + \sum_m \alpha_m \ln y_m + \frac{1}{2}\sum_m\sum_n \alpha_{mn} \ln y_m \ln y_n + \sum_{j=2} \beta_j \ln \hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2} \beta_{jk} \ln \hat{x}_j \ln \hat{x}_k \\ &+ \sum_m\sum_{j=2} \delta_{mj} \ln \hat{x}_j \ln y_m + u_h\end{aligned} \quad (20)$$

where $u_h = \ln\lambda\big\{1 + \big[\sum_m \alpha_m\big] + \sum_m \big[\sum_n \alpha_{mn}\big] \ln y_m + \big[\sum_j \delta_{mj}\big] \ln \hat{x}_j\big\} + \frac{1}{2}\sum_m\sum_n \alpha_{mn}\{\ln\lambda\}^2$. It is clear from the above that u_h is related to $\ln\lambda$ in a highly nonlinear fashion.[7] However, since (19) and (20) are identical, their inefficiencies are also the same. That is, $-\ln\theta$ in (19) is the same as u_h in (20). Thus, the estimated values of input-oriented inefficiency $\ln\theta$ from (19) can be used to estimate hyperbolic inefficiency

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[6] Note that these identifying/normalizing constraints make the transformation function homogeneous of degree one in inputs. In the efficiency literature one defines the IDF as the distance (transformation) function which is homogeneous of degree one in inputs. Here we view the homogeneity property as identifying restrictions on the transformation function without using the notion of a distance function.

[7] This relationship is similar to the relationship between input- and output-oriented technical inefficiency, estimation of which is discussed in details in Kumbhakar and Tsionas (2006).

$\ln\lambda$ by solving the quadratic equation $-\ln\theta = \ln\lambda\big\{1+\big[\sum_m \alpha_m\big]+\sum_m\big[\sum_n \alpha_{mn}\big]\ln y_m+\big[\sum_j \delta_{mj}\big]\ln\hat{x}_j\big\}+\frac{1}{2}\sum_m\sum_n \alpha_{mn}\{\ln\lambda\}^2$.

Finally, if we use the following normalizations $\beta_1 = -1, \beta_{1j} = 0, \forall\, j, \delta_{m1} = 0, \forall\, m, \lambda = 1$, the input requirement function is obtained, which can be written as

$$\begin{aligned}\textbf{TL IRF:}\quad \ln x_1 = {} & \alpha_0 + \sum_{j=2}\beta_j \ln x_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln x_j \ln x_k + \sum_m \alpha_m \ln y_m \\ & + \frac{1}{2}\sum_m\sum_n \alpha_{mn}\ln y_m \ln y_n + \sum_m\sum_{j=2}\delta_{mj}\ln x_j \ln y_m + u \qquad (21)\end{aligned}$$

where $u = \ln\theta(-1+\sum_{j=2}\beta_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln x_j + \sum_m\sum_{j=2}\delta_{mj}\ln y_m) + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}(\ln\theta)^2$.

It is clear from the above that starting from the translog transformation function specification in (15) one can derive the production function, the output and input distance functions, and the input requirement function simply by using different normalizations. Furthermore, these formulations show how technical inefficiency transmits from one specification into another. Note that other than the input and output distance functions, technical inefficiency appears in a very complicated form. So although all these specifications are algebraically the same, the question that naturally arises is which formulation is easier to estimate, and whether a particular specification has fewer endogenous regressors and is preferred to other specifications.

3 Estimation: A single equation approach

The main issue in estimating any of the primal functions we introduced in the preceding section is endogeneity of the right hand side variables. Since the right hand side variables differ depending on the specification one uses, we now discuss the endogeneity of the regressors in each specification.

Before discussing estimation issues, it is worth examining the sources of inefficiency (from now on we will label it as u, which is related to $\ln\lambda$ and/or $\ln\theta$). If one argues, following Hoch (1955, 1962), Mundlak and Hoch (1965), Mundlak (1961) that u reflects 'management' and other unobserved variables and assume that producers know them fully, then u would affect input demand and output supply. That is, input and output quantities will be correlated with u.[8] This will make OLS/ML estimators inconsistent. We follow a similar line of argument for inefficiency which is assumed to be known to the producers, and examine possible solutions to this endogeneity problem from economic behavior of the producers. We discuss the CET-CD and the translog cases in separate sub-sections.

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[8] See also Olley and Pakes (1996) and Levinsohn and Petrin (2003) on the solution to the endogeneity problem using panel data without using any behavioral assumption.

3.1 The CET-CD technology

First, we consider the CET-CD technology with the ODF formulation in (10). We introduce stochastic element into the model via A, by specifying it as $\ln A = \alpha_0 + v$ where $v \sim i.i.d.(0, \sigma_v^2)$ is production uncertainty (unknown external shock). The idea is to check whether the regressors are correlated with u and v. For this we assume that producers maximize expected profit[9] subject to the technology $A\, g(\lambda y)\, f(\theta x) = 1$ where $g(\lambda y)$ and $f(\theta x)$ are CET and CD functions. The FOCs of expected profit maximization are[10]

$$\begin{aligned} p_m/p_1 &= \{\delta_m/\delta_1\}\{y_m/y_1\}^{(c-1)} \\ w_j/w_1 &= \{\beta_j/\beta_1\}\{x_1/x_j\} \\ w_1/p_1 &= -\{\beta_1/\delta_1\}\{y_1/x_1\}\left[\delta_1 + \sum \delta_m (y_m/y_1)^c\right] \end{aligned} \tag{22}$$

where p_m and w_j are prices of y_m and x_j. The first $(M-1)$ equations in (22) can be solved for y_m/y_1, the second $(J-1)$ equations can be solved for x_j/x_1 in terms of price ratios. Since u and v do not appear in any of these equations the output and input ratios will be uncorrelated with u and v. Therefore we can treat them as exogenous. Finally, the last equation in (22) can be solved for y_1/x_1 (after replacing the solutions of output ratios (from the first $M-1$ equations) which are not correlated with u and v). Since u and v do not appear in this equation the solution of y_1/x_1 will be independent of them. Note that solutions of y_m and x_j (for which we need to use relationship in (9) or in (10) which has u and v in it) will depend on both u and v. Thus, if one wants to estimate the ODF in (10) endogeneity of inputs (which do not appear in ratio forms) has to be taken care of.

However, this endogeneity problem can be avoided completely if we rewrite (10) as

$$(1+r)\ln y_1 + \alpha_0 + (1/c)\ln\left[\delta_1 + \sum \delta_m (y_m/y_1)^c\right] + \sum_j \beta_j \ln(x_j/x_1) + r\, \ln(x_1/y_1) + u + v = 0 \tag{23}$$

when $r = \sum_j \beta_j$. Note that in this equation the only endogenous variable (correlated with u and v) is $\ln y_1$. All other variables appear in ratio forms and are uncorrelated with u and v which follow from the FOCs in (22). Thus, one can use the standard stochastic frontier techniques to estimate the model in (23) after making distributional assumptions on u and v.

If one wants to use the IDF the same problem appears. The $\ln y_m$ variables in (11) are endogenous

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[9] For more on this see Zellner et al. (1966) and Coelli (2000).

[10] These FOCs are the same irrespective of whether v is included or not, known or unknown. Because of this, it does not matter whether v is ignored to start with and then it is added at the estimation stage. This is, however, not true for other functions such as the translog, as we will see later.

(correlated with u and v) and the IDF cannot be used to estimate the parameters consistently. As before this endogeneity problem can be resolved by rewriting the IDF in (11) in the following form:

$$\ln x_1(1+r) + \alpha_0 + (1/c)\ln\left[\delta_1 + \sum_{m=2}\delta_m(y_m/y_1)^c\right] + \sum_{j=2}\beta_j \ln(x_j/x_1) + \ln(y_1/x_1) + u + v \qquad (24)$$

Note that (24) has only one endogenous variable ($\ln x_1$) and it can be used as the dependent variable. All other variables are in ratio form and are independent of u and v. Note though that (24) is not the standard IDF.

The same conclusion is reached if one uses the hyperbolic IDF in (12). That is, since the y_m variables do not appear in ratio form, the hyperbolic IDF will have the same inconsistency problem as the standard IDF. Finally, estimation of the IRF in (14) is the worst in the sense that all the regressors are endogenous, and it would require instruments for all.

Based on the above discussions we come to the conclusion that if both outputs and inputs are decision variables and u is fully known to the producers, one cannot use the standard IRF, IDF and ODF (which is the same as the production function) to estimate the parameters consistently. The IDF (ODF) is appropriate to use only when outputs (inputs) are exogenous in which case profit maximization hypothesis will be equivalent to cost minimization (revenue maximization). The other alternative is to express all the regressors in ratio forms irrespective of whether an input or an output distance function is used. In the CET-CD model this strategy is the best because in these specifications all the regressors are exogenous (i.e., there is only one endogenous variable in the model). This is exactly what we need in order to estimate efficiency using the stochastic frontier (SF) technique, which requires the regressors to be independent of u and v.

3.2 Translog functions

3.2.1 Production and distance functions

We now discuss these issues in the context of a translog transformation function. Translog is more flexible than the CET-CD model and is linear in parameters whereas the CET-CD model is nonlinear. To start with we assume that $\ln A = \alpha_0 + v$ but profit maximization takes place at $E(v) = 0$ so that the FOCs are not affected by v, although it will affect the input demand and output supply functions.[11]

Consider the production function formulation in (16) (based on the translog transformation function) first. It has serious endogeneity problem since all the regressors (inputs and outputs) are endogenous (correlated with u and v). The same problem arises in estimating the IRF in (21) because all the regressors

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[11] We consider the expected profit maximization case with optimization errors in a separate section.

are endogenous. An additional problem with both the production and input requirement functions is that the results of these models are not invariant to the choice of the dependent variable. Because of these problems we do not discuss estimation of these models.

To show whether we have similar problems in estimating the IDF, ODF, and the hyperbolic IDF, we consider the FOCs of profit maximization, which are: $p_m + \Lambda\, f_{y^*_m}(y^*, x^*)\, \lambda\, A = 0$ and $-w_j + \Lambda f_{x^*_j}(y^*, x^*)\, \theta\, A = 0$ where $f_{y^*_m}(.)$ and $f_{x^*_j}(.)$ are partial derivatives of $f(y^*, x^*)$ with respect to y^*_m and x^*_j respectively, and Λ is the Lagrange multiplier. These FOCs, along with translog transformation function in (15), can be solved (after using some normalizing constraints to identify the parameters as well as technical inefficiency) for y_m and x_j which will depend on prices and u and v. That is, all the inputs and outputs will be correlated with inefficiency.

One can rewrite the above FOCs as

$$\begin{aligned} w_j x_j / w_1 x_1 &= \partial \ln f(y^*, x^*)/\partial \ln x^*_j \div \partial \ln f(y^*, x^*)/\partial \ln x^*_1,\ j = 2, \cdots, J \\ p_m y_m / p_1 y_1 &= \partial \ln f(y^*, x^*)/\partial \ln y^*_m \div \partial \ln f(y^*, x^*)/\partial \ln y^*_1,\ m = 2, \cdots, M \\ w_1 x_1 / p_1 y_1 &= -\,\partial \ln f(y^*, x^*)/\partial \ln x^*_1 \div \partial \ln f(y^*, x^*)/\partial \ln y^*_1 \end{aligned}$$

and use the translog transformation function in (15) together with various normalizing constraints to check whether the regressors in various specification we considered are exogenous. First, we consider the ODF specification in (17) for which the above FOCs can be written as

$$\begin{aligned} \frac{p_m}{p_1}\hat{y}_m &= \frac{\alpha_m + \sum_n \alpha_{mn} \ln \hat{y}_n + \sum_j \delta_{mj} \ln x_j}{\alpha_1 + \sum_n \alpha_{1n} \ln \hat{y}_n + \sum_j \delta_{1j} \ln x_j} \\ \frac{w_j}{w_1}\frac{x_j}{x_1} &= \frac{\beta_j + \sum_k \beta_{jk} \ln x_k + \sum_m \delta_{mj} \ln \hat{y}_m}{\beta_1 + \sum_k \beta_{1k} \ln x_k + \sum_m \delta_{m1} \ln \hat{y}_m} \\ \frac{w_1}{p_1}\frac{x_1}{y_1} &= -\frac{\beta_1 + \sum_k \beta_{1k} \ln x_k + \sum_m \delta_{m1} \ln \hat{y}_m}{\alpha_1 + \sum_n \alpha_{1n} \ln \hat{y}_n + \sum_j \delta_{1j} \ln x_j} \end{aligned} \tag{25}$$

These FOCs along with the ODF in (17) can be solved to derive the input demand and output supply functions which will depend on u and v and the price variables. Here the idea is to check whether these $M-1+J-1+1 = M+J-1$ equations can be solved for $\hat{y}_m, x_j, m = 2, \cdots, M; j = 1, \cdots, J$ and whether solutions of $\hat{y}_m$ and x_j (which are the regressors in (17)) can be treated as exogenous. Although we have $M+J-1$ equations in (25) it is not possible to solve them for $\hat{y}_m$ and x_j since the last equation has y_1 which increases number of endogenous variables to $M+J$. Furthermore, since the output ratios in the first $M-1$ equations are functions of x_j which are endogenous, the standard argument that output ratios are exogenous in ODF does not hold and therefore parameter estimates will be inconsistent resulting in inappropriate estimates of inefficiency.[12] The exceptions are: (i) the inputs are exogenous which means the first $(M-1)$ FOCs can be solved for $\hat{y}_m$ in terms of output price ratios and $\ln x_j$, and (ii) the technology is homogeneous (constant returns to scale).

Given that neither output ratios nor input quantities are exogenous, the question is whether one needs to use instruments for all the regressors. If we rewrite the above FOCs in terms of both input and output ratios, viz.,

$$\begin{aligned} \frac{p_m}{p_1}\hat{y}_m &= \frac{\alpha_m + \sum_n \alpha_{mn}\ln\hat{y}_n + \sum_j \delta_{mj}\ln\hat{x}_j + (\sum_j \delta_{mj})\ln x_1}{\alpha_1 + \sum_n \alpha_{1n}\ln\hat{y}_n + \sum_j \delta_{1j}\ln\hat{x}_j + (\sum_j \delta_{1j})\ln x_1} \\ \frac{w_j}{w_1}\hat{x}_j &= \frac{\beta_j + \sum_k \beta_{jk}\ln\hat{x}_k + \sum_m \delta_{mj}\ln\hat{y}_m + (\sum_k \beta_{jk})\ln x_1}{\beta_1 + \sum_k \beta_{1k}\ln\hat{x}_k + \sum_m \delta_{m1}\ln\hat{y}_m + (\sum_k \beta_{1k})\ln x_1} \end{aligned} \tag{26}$$

then we can solve for $\hat{y}_m, \hat{x}_j, m = 2, \cdots, M; j = 2, \cdots, J$ in terms of price ratios and x_1. These solutions are not entirely exogenous because of the presence of one endogenous variable, x_1. To check whether this helps in reducing number of endogenous variables in the ODF in (17), we rewrite it as

$$\begin{aligned} \ln y_1 = {}& \alpha_0 + \sum_{j=2}\beta_j \ln\hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln\hat{x}_j\ln\hat{x}_k + \sum_{m=2}\alpha_m\ln\hat{y}_m + \frac{1}{2}\sum_{m=2}\sum_{n=2}\alpha_{mn}\ln\hat{y}_m\ln\hat{y}_n \\ &+ \sum_{j=2}\sum_{m=2}\delta_{mj}\ln\hat{x}_j\ln\hat{y}_m + \ln x_1\Big\{\sum_{j=2}\big[\beta_j + (\sum_k \beta_{jk})\ln\hat{x}_j + \sum_{m=2}\delta_{mj}\ln\hat{y}_m\big]\Big\} \\ &+ \{\ln x_1\}^2\frac{1}{2}\sum_j\sum_k \beta_{jk} + u + v \end{aligned} \tag{27}$$

Given that $\hat{y}_m$ and $\hat{x}_j$ can be solved in terms of exogenous variables (price ratios) and one endogenous variable ($\ln x_1$), we need to deal with endogeneity of $\ln x_1$ only. That is, if we rewrite the ODF in ratio form (as in (27)) and use instrument for $\ln x_1$ the inconsistency problem can be avoided. However, note that the ODF rewritten in ratio form is different from the standard ODF that is used in the literature.

If we impose constant returns to scale (CRTS) assumption[13] in (17), it will give these additional restrictions, viz., $\sum_j \delta_{mj} = 0, \forall m$ and $\sum_k \beta_{jk} = 0, \forall j$ (which implies $\sum_j \beta_{jk} = 0, \forall k$ because of symmetry).

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[12] See Coelli (2000) for a discussion of some of these issues.

[13] Returns to scale in terms of the transformation function is defined as $\sum_j \partial\ln f(\cdot)/\partial\ln x_j \div \sum_m \partial\ln f(\cdot)/\partial\ln y_m$ (Panzar and Willig (1977)).

With these additional restrictions returns to scale is $\sum_j \beta_j$ and the ODF in (27) can be expressed as

$$\begin{aligned}\ln y_1 = \alpha_0 + (\sum_j \beta_j)\ln x_1 + \sum_{j=2}\beta_j \ln \hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln \hat{x}_j \ln \hat{x}_k + \sum_{m=2}\alpha_m \ln \hat{y}_m \\ + \frac{1}{2}\sum_{m=2}\sum_{n=2}\alpha_{mn}\ln \hat{y}_m \ln \hat{y}_n + \sum_{j=2}\sum_{m=2}\delta_{mj}\ln \hat{x}_j \ln \hat{y}_m - \ln \lambda + v\end{aligned} \tag{28}$$

and the FOCs in (26) become

$$\begin{aligned}\frac{p_m}{p_1}\hat{y}_m &= \frac{\alpha_m + \sum_n \alpha_{mn}\ln \hat{y}_n + \sum_j \delta_{mj}\ln \hat{x}_j}{\alpha_1 + \sum_n \alpha_{1n}\ln \hat{y}_n + \sum_j \delta_{1j}\ln \hat{x}_j} \\ \frac{w_j}{w_1}\hat{x}_j &= \frac{\beta_j + \sum_k \beta_{jk}\ln \hat{x}_k + \sum_m \delta_{mj}\ln \hat{y}_m}{\beta_1 + \sum_k \beta_{1k}\ln \hat{x}_k + \sum_m \delta_{m1}\ln \hat{y}_m}\end{aligned} \tag{29}$$

where $\hat{x}_j = x_j/x_1$. It is clear from these FOCs that input and output ratios can be solved in terms of input and output prices (which are exogenous). Thus, with the constant CRTS assumption, the input and output ratios can be treated as exogenous. Furthermore, $\ln(x_1/y_1)$ will be also exogenous because we can add another equation in the above FOCs, viz.,

$$-\frac{w_1}{p_1}\frac{x_1}{y_1} = \frac{\beta_1 + \sum_k \beta_{1k}\ln \hat{x}_k + \sum_m \delta_{m1}\ln \hat{y}_m}{\alpha_1 + \sum_n \alpha_{1n}\ln \hat{y}_n + \sum_j \delta_{1j}\ln \hat{x}_j}$$

to solve for x_1/y_1. Thus all the regressors in (28) are exogenous, except for $\ln x_1$. Since only $\ln x_1$ is endogenous, one needs to find instrument(s) for $\ln x_1$ only. This benefit comes from the CRTS assumption.

Instead of using the CRTS assumption, assume that the transformation function is separable which in the present case means $\delta_{mj} = 0, \forall\, m, j$. This assumption enables us to solve for $\hat{y}_m$ from the first $(M-1)$ equations in (25) in terms of output prices. Thus, output ratios will be exogenous. However, the input ratios will be endogenous because their solutions from the last $(J-1)$ equations in (25) will depend on x_1 which is endogenous. Thus to estimate the ODF one needs to take care of endogeneity of all the inputs (i.e., instruments for all the input variables are required), even when separability is assumed.

Coming back to the CRTS case the endogeneity problem can be resolved if $\ln x_1$ is expressed in the ratio form (i.e., in terms of $\ln(x_1/y_1)$). For this we rewrite (28) as

$$\begin{aligned}\ln y_1 = \{1/(1-\sum_j \beta_j)\}\Big\{\alpha_0 + \{\sum_j \beta_j\}\ln(x_1/y_1) + \sum_j \beta_{j=2}\ln \hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln \hat{x}_j \ln \hat{x}_k \\ + \sum_{m=2}\alpha_m \ln \hat{y}_m + \frac{1}{2}\sum_{m=2}\sum_{n=2}\alpha_{mn}\ln \hat{y}_m \ln \hat{y}_n + \sum_{j=2}\sum_{m=2}\delta_{mj}\ln \hat{x}_j \ln \hat{y}_m\Big\} + u + \nu,\end{aligned} \tag{30}$$

where $u = -\{1/(1-\sum_j \beta_j)\}\ln\lambda$ and $\nu = \{v/(1-\sum_j \beta_j)\}$. Note that all the regressors in the equation in (30) are exogenous. However, (30) is not a standard ODF. But the good thing is that one can use the standard SF technique to estimate the parameters as well as inefficiency.

If we further restrict returns to scale and make it unity, then $\sum_j \beta_j = 1$ which helps us to rewrite (28) as

$$\begin{aligned}\ln(y_1/x_1) &= \alpha_0 + \sum_{j=2}\beta_j \ln\hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln\hat{x}_j\ln\hat{x}_k + \sum_{m=2}\alpha_m \ln\hat{y}_m \\ &+ \frac{1}{2}\sum_{m=2}\sum_{n=2}\alpha_{mn}\ln\hat{y}_m\ln\hat{y}_n + \sum_{j=2}\sum_{m=2}\delta_{mj}\ln\hat{x}_j\ln\hat{y}_m - \ln\lambda + v \end{aligned} \tag{31}$$

Note that all the regressors are in ratios which are exogenous (as argued from the FOCs in (29)). Although (31) is not the standard ODF, it helps resolving the endogeneity problem, and therefore paves the way for using the standard SF methodology. However, unitary RTS is not consistent with profit maximizing behavior because it violates the second-order conditions of profit maximization.

We can use the exact same arguments to show that the regressors in the IDF in (19) are endogenous unless (i) outputs are exogenous, (ii) the technology is homogeneous (CRTS) which implies these additional constraints, viz., $\sum_m \delta_{mj} = 0, \forall j$ and $\sum_k \beta_{jk} = 0, \forall j$. Under CRTS the IDF in (19) can be written in such a way that the outputs are in ratio form except one (i.e., y_1). That is, the resulting IDF will have one endogenous variable as regressor ($\ln y_1$). This is similar to the ODF result in (28) where $\ln x_1$ is the only endogenous regressor. Instead of CRTS assumption, if we assume the transformation function to be separable, then all the output variables in the IDF model will be endogenous. This is the same as the ODF model result.

Under CRTS the endogeneity problem in the IDF can be resolved if $\ln y_1$, the only endogenous regressor, is expressed in the ratio form (i.e., $\ln(x_1/y_1)$). For this we rewrite the IDF in (19) as

$$\begin{aligned}\ln x_1 = \{1/(1-\sum_m \alpha_m)\}\Big\{&\alpha_0 - (\sum_m \alpha_m)\ln(x_1/y_1) + \sum_{j=2}\beta_j\ln\hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln\hat{x}_j\ln\hat{x}_k \\ &+ \sum_{m=2}\alpha_m\ln\hat{y}_m + \frac{1}{2}\sum_{m=2}\sum_{n=2}\alpha_{mn}\ln\hat{y}_m\ln\hat{y}_n + \sum_{m=2}\sum_{j=2}\delta_{mj}\ln\hat{x}_j\ln\hat{y}_m\Big\} + u + \nu,\end{aligned} \tag{32}$$

where $u = -\ln\theta\{1/(1-\sum_m \alpha_m)\}$ and $\nu = v/(1-\sum_m \alpha_m)$. Note that although the equation in (32) is not a standard IDF all the regressors in it are exogenous and the standard frontier technique can be used to estimate the parameters and inefficiency in a single step.

If we assume RTS to be unity which adds the additional constraint $\sum_m \alpha_m = 1$, the IDF in (19) can be expressed as

$$\ln(x_1/y_1) = \alpha_0 + \sum_{j=2} \beta_j \ln \hat{x}_j + \frac{1}{2} \sum_{j=2} \sum_{k=2} \beta_{jk} \ln \hat{x}_j \ln \hat{x}_k + \sum_{m=2} \alpha_m \ln \hat{y}_m$$
$$+ \frac{1}{2} \sum_{m=2} \sum_{n=2} \alpha_{mn} \ln \hat{y}_m \ln \hat{y}_n + \sum_{m=2} \sum_{j=2} \delta_{mj} \ln \hat{x}_j \ln \hat{y}_m + u + v, \quad (33)$$

where $u = -\ln \theta$. Note that although this is not a standard IDF but it can be estimated using the standard frontier technique since the regressors are exogenous. The downside is that unitary RTS is inconsistent with profit maximizing behavior.

In summary we conclude that even under constant and unitary RTS one cannot use the standard IDF (ODF) models when both inputs and outputs are endogenous. However, the IDF (ODF) models can be written in a way that makes all the regressors exogenous. These specifications are useful because the use of SF techniques require the regressors to be independent of u and v.

If RTS is non-constant, all the regressors will be correlated with the composed error term, $u + v$. This is true for IDF, ODF, production and input requirement functions. Even if the technology is separable, input (output) ratios in the IDF (ODF) are exogenous but outputs (inputs) are endogenous. Thus the common feature in all these (non-CRTS) models is that some of the regressors are endogenous and therefore a single-step SF technique cannot be used. One has to use a two-step approach in which the first step is to estimate the model using IV for the endogenous variables. The residuals of the estimating equation (IDF or ODF) can then be used in the second stage to estimate inefficiency using the SF technique. The advantage of this two-step approach is that it is easy to estimate and the estimated parameters in the first-stage do not depend on distributional assumptions. The downside is that the estimators are not efficient.

4 Estimation: A system approach

4.1 The CET-CD system

Here the production system consists of the following equations

$$
\begin{aligned}
-\ln y_1 &= \alpha_0 + (1/c)\ln\left[\delta_1 + \sum_m \delta_m (y_m/y_1)^c\right] + \sum_j \beta_j \ln x_j + u + v \\
\ln[p_m/p_1] &= \ln[\delta_m/\delta_1] + (c-1)\ln[y_m/y_1] + \varepsilon_m \\
\ln[w_j/w_1] &= \ln[\beta_j/\beta_1] + \ln x_1 - \ln x_j + v_j \\
\ln[w_1/p_1] &= \ln[-\beta_1/\delta_1] + \ln y_1 - \ln x_1 + \ln\left[\delta_1 + \sum_m \delta_m (y_m/y_1)^c\right] + v_1 \qquad (34)
\end{aligned}
$$

where $\varepsilon_m (m = 2, \cdots, M), v_j (j = 1, \cdots, J)$ and v_1 are optimization errors, v is the uncertainty component in the production function. Note that the number of equations $(M + J)$ is exactly the same as number of endogenous variables $(\ln y_1, \ln(y_m/y_1), \ln x_j; m = 2, \cdots, M; j = 1, \cdots, J)$.

Assuming a sample of N firms, we need to find the probability density function (pdf) of the error vector $(u_i + \eta_{1i}, \eta_{2i}, \ldots, \eta_{Ji})'$, where the subscript i denotes firm $(i = 1, \ldots, N)$ and $\eta_i' = (v_i, \varepsilon_{mi}, v_{ji}), j = 1, \ldots, J; m = 2, \ldots, M$. Define $Z_i = (b\, u_i + \eta_i)'$ where $b = (1, 0, \ldots, 0)'$.

Here we follow the standard practice in the stochastic frontier literature and assume u_i to be distributed as half-normal (i.e., $u_i \sim$ *i.i.d.* $N(0, \sigma_u^2)$ truncated at zero from below, $u \geq 0$). Similarly, $\eta_i = (\eta_{1i}, \ldots, \eta_{Ji})'$ is assumed to be distributed as *i.i.d.* $N(0, \Sigma)$. The elements of η_i are assumed to be independent of u_i. Since both u_i and η_i are *i.i.d.* across firms, we drop the firm subscript in the following derivation.

The pdf of Z, $f(Z)$, can be expressed as

$$f(Z) = \int_0^\infty f(Z, u)du = \int_0^\infty f(Z|u)h(u)du$$

where $f(Z, u)$ is the joint pdf of Z and u, and $h(u)$ is the pdf of u. Using the distributional assumptions on u and η, the above integral can be expressed as (Kumbhakar (2001))[14]

$$
\begin{aligned}
f(Z) &= \frac{2}{(2\pi)^{(J+1)/2}|\Sigma|^{1/2}\sigma_u} \int_0^\infty \exp\Big\{-\frac{1}{2}[(Z - b\,u)'\Sigma^{-1}(Z - b\,u) + u^2/\sigma_u^2]\Big\}du \\
&= \frac{2\,\sigma\, \exp(-a/2)}{(2\pi)^{(J/2)}|\Sigma|^{1/2}\sigma_u}\Phi(Z'\Sigma^{-1}b\,\sigma) \qquad (35)
\end{aligned}
$$

---

[14] Note that in Kumbhakar (2001) $u \leq 0$ whereas here we have $u \geq 0$. This changed the sign inside $\Phi(.)$.

where $\sigma^2 = (1/\sigma_u^2 + b'\Sigma^{-1}b)^{-1}$, $a = Z'\Sigma^{-1}Z - \sigma^2(Z'\Sigma^{-1}b)^2$, and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable.

Based on the above distributional assumptions on u and η, the log-likelihood function for a sample of N firms (firm subscript i is added) can then be written as

$$\mathcal{L} = \text{constant} - (N/2)\ \ln|\Sigma| + N\ln\sigma + \sum_i \Phi(Z_i'\Sigma^{-1}b\,\sigma) - N\ln\sigma_u(1/2)\sum_i a_i + \sum_i |D_i|, \tag{36}$$

where D_i is the Jacobian (matrix) of the transformation from Z_i to $\ln y_m, \ln x_j, m = 1, \ldots, M; j = 1, \ldots, J$. All other parameters are defined before. Maximization of the above log-likelihood function gives consistent estimates of the parameters of the production function as well as σ_u^2 and those in Σ.

After obtaining estimates of the parameters, technical inefficiency for each firm can be obtained using the decomposition formula in Kumbhakar (1987, 1996, 2001), which is a generalization of the Jondrow et al. (1982) result to a simultaneous equation system. This formula is based on the conditional mean or mode of u given $(b\,u + \eta)$. Since the conditional distribution of u given $(b\,u + \eta)$ is $N(Z'\Sigma^{-1}b\sigma^2, \sigma^2)$ truncated at zero from above, the following estimator of u is suggested (Kumbhakar (2001))

$$\hat{u} = E(u|(b\,u + \eta)) = \mu - \sigma\frac{\phi(\mu/\sigma)}{\Phi(\mu/\sigma)} \tag{37}$$

where $\phi(\cdot)$ is the pdf of a standard normal variable and $\mu = Z'\Sigma^{-1}b\,\sigma^2$.

Since endogeneity of all inputs and outputs are explicitly considered, the above system will be the same no matter whether the production, IDF or ODF is used. That is, in a system approach there is no advantage of using one specification over another, and one does not have to worry about whether to use a production or a distance function. An alternative is to use a system approach (system IV, system GMM, iterated nonlinear 3SLS, etc.) ignoring technical inefficiency in the fist step, and then estimate technical inefficiency from the residuals of the production function in the second step using the standard SF technique. Note that if inefficiency distribution has constant mean and variance, only the intercept term in the transformation function will be biased when inefficiency is ignored in the first step. The intercept is estimated in the second step along with inefficiency using the residuals from the transformation function in the standard Jondrow et al. (1982) formula. This approach might be preferred to the single-step MLE applied to the system because it is simple to use and the parameters in the first step are not affected by distributional assumptions. Distributional assumptions are used only in the second step to estimate inefficiency.

4.2 The TL system

In the translog case the inefficiency term appears linearly in the IDF and ODF specifications. Because of this we consider only the distance function specifications. First, we start from the ODF specification in (17). The the corresponding FOCs with optimization error (allocative inefficiency) can be written as

$$\begin{aligned}
\frac{p_m}{p_1}\,\hat{y}_m \exp(\varepsilon_m) &= \frac{\alpha_m + \sum_n \alpha_{mn} \ln \hat{y}_n + \sum_j \delta_{mj} \ln x_j}{\alpha_1 + \sum_n \alpha_{1n} \ln \hat{y}_n + \sum_j \delta_{1j} \ln x_j} \\
\frac{w_j}{w_1}\,\frac{x_j}{x_1} \exp(v_j) &= \frac{\beta_j + \sum_k \beta_{jk} \ln x_k + \sum_m \delta_{mj} \ln \hat{y}_m}{\beta_1 + \sum_k \beta_{1k} \ln x_k + \sum_m \delta_{m1} \ln \hat{y}_m} \\
\frac{w_1}{p_1}\,\frac{x_1}{y_1} \exp(v_1) &= -\frac{\beta_1 + \sum_k \beta_{1k} \ln x_k + \sum_m \delta_{m1} \ln \hat{y}_m}{\alpha_1 + \sum_n \alpha_{1n} \ln \hat{y}_n + \sum_j \delta_{1j} \ln x_j}
\end{aligned} \tag{38}$$

where $\varepsilon_m, m = 2, \cdots, M$ and $v_j, j = 1, \cdots, J$ are optimization errors. The ODF in (17), after adding the v term, along with the above FOCs (after taking log) define a system of $M + J$ equations with $M + J$ endogenous variables (x_j, $j = 1, \cdots, J$, y_1 and $\hat{y}_m, m = 2, \cdots, M$). Similar to the CET-CD system, the log-likelihood function for the present system (after taking log of the FOCs) can be derived based on the distributional assumptions on u and η (defined for the CET-CD case). Maximization of the log-likelihood function will give consistent estimators of the parameters which can then be used to obtain firm-specific estimate of output-oriented technical inefficiency from the formula in (37). Since the ML method is quite complicated, one can use the simpler alternative, viz., a two-step procedure. In the first step, technical inefficiency is ignored and the system is estimated using iterative 3SLS or a system GMM in which log of price ratios, their squares and cross-products can be used as instruments. Once the parameters are estimated, residuals from the ODF can be used in the second step to estimate the intercept and technical inefficiency using MLE. Note that if inefficiency distribution has constant mean and variance (which is true for half-normal, truncated normal, exponential, gamma distributions) only the intercept term will be biased if inefficiency is ignored. The intercept is estimated in the second step along with inefficiency. As noted above this two-step procedure is easy to use and the parameters of the ODF are invariant to the distributional assumptions on the error components. Distributional assumptions are only used to estimate inefficiency in the second step.

If one stars from the IDF in (19) the corresponding FOCs can be written as

$$\begin{aligned}
\frac{p_m}{p_1}\frac{y_m}{y_1}\exp(\varepsilon_m) &= \frac{\alpha_m + \sum_n \alpha_{mn}\ln y_n + \sum_j \delta_{mj}\ln \hat{x}_j}{\alpha_1 + \sum_n \alpha_{1n}\ln y_n + \sum_j \delta_{1j}\ln \hat{x}_j} \\
\frac{w_j}{w_1}\hat{x}_j\exp(v_j) &= \frac{\beta_j + \sum_k \beta_{jk}\ln \hat{x}_k + \sum_m \delta_{mj}\ln y_m}{\beta_1 + \sum_k \beta_{1k}\ln \hat{x}_k + \sum_m \delta_{m1}\ln y_m} \\
\frac{w_1}{p_1}\frac{x_1}{y_1}\exp(v_1) &= -\frac{\beta_1 + \sum_k \beta_{1k}\ln \hat{x}_k + \sum_m \delta_{m1}\ln y_m}{\alpha_1 + \sum_n \alpha_{1n}\ln y_n + \sum_j \delta_{1j}\ln \hat{x}_j}
\end{aligned} \tag{39}$$

where $\varepsilon_m, m = 2, \cdots, M$ and $v_j, j = 1, \cdots, J$ are optimization errors. The IDF in (19) along with the above FOCs (after taking log) define the system[15] with $M + J$ endogenous variables ($\hat{x}_j$, $j = 2, \cdots, J$, x_1 and $y_m, m = 1, \cdots, M$) which can be estimated using MLE. This requires distributional assumptions on the error components. Again the formula in (37) can then be used to estimate input-oriented technical inefficiency.

Thus the difference between the two systems is that in the former output-oriented technical inefficiency is directly estimated while in the latter input-oriented technical inefficiency is estimated. However, it is not necessary to estimate both. It is possible to estimate input-oriented technical inefficiency ($\ln\theta$) from the ODF and vice versa. This requires solving a quadratic equation, viz., $-\ln\lambda + \ln\theta\sum_j\{\beta_j + \sum_k \beta_{jk}\ln x_k + \sum_m \delta_{mj}\ln y_m\} + \frac{1}{2}(\sum_j\sum_k \beta_{jk})(\ln\theta)^2 = 0$ for $\ln\theta$ using estimated values of the parameters, $\ln\lambda$ and data.

5 Estimation of translog models with production uncertainty

As before we introduce stochastic element into the transformation function by assuming $\ln A = \alpha_0 + v$ where $v \sim i.i.d.(0, \sigma_v^2)$ represents production uncertainty. Since v is not known, following Zellner et al. (1966), we assume that producers maximize expected profit. To avoid repetitions here we consider only the translog IDF and ODF models.

5.1 The ODF model

Here the problem is to maximize $E(\pi|u)$ subject to the ODF

$$\begin{aligned}
\ln y_1 = \alpha_0 &+ \sum_j \beta_j \ln x_j + \frac{1}{2}\sum_j\sum_k \beta_{jk}\ln x_j \ln x_k + \sum_{m=2}\alpha_m \ln \hat{y}_m \\
&+ \frac{1}{2}\sum_{m=2}\sum_{n=2}\alpha_{mn}\ln\hat{y}_m \ln\hat{y}_n + \sum_{j=2}\sum_m \delta_{mj}\ln x_j \ln \hat{y}_m + u + v
\end{aligned} \tag{40}$$

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[15] See Karagiannis et al. (2006) and Coelli et al. (2008) for a system using translog input distance function based on cost minimization behavior.

where $\pi = p'y - w'x$. Rewrite it as $y_1 = y_1^e e^v/\mu$ where $y_1^e = E(y_1|u)$ and $\mu = E(e^v)$ so that we can write $E(\pi) = p_1 y_1^e + \sum_{m=2} p_m y_m - \sum_j w_j x_j$. The FOCs with respect to $y_m, m = 2, \cdots, M$ and $x_j, j = 1, \cdots, J$ can be written as

$$\begin{aligned} -\frac{\partial \ln y_1}{\partial \ln \hat{y}_m} &= \frac{p_m}{p_1}\frac{y_m}{y_1}\frac{1}{\mu}\exp(\xi_m + v) = -[\alpha_m + \sum_n \alpha_{mn} \ln \hat{y}_n + \sum_j \delta_{mj} \ln x_j] \\ \frac{\partial \ln y_1}{\partial \ln x_j} &= \frac{w_j}{p_1}\frac{x_j}{y_1}\frac{1}{\mu}\exp(\zeta_j + v) = \beta_j + \sum_k \beta_{jk} \ln x_k + \sum_m \delta_{mj} \ln \hat{y}_m \end{aligned} \tag{41}$$

where ξ_m and ζ_j are optimization errors with respect to y_m and x_j, respectively. Logarithms of these $(M - 1 + J)$ FOCs along with the ODF in (40) define the system in which the endogenous variables are $\ln \hat{y}_m, \ln y_1$ and $\ln x_j$. Note that the production uncertainty term v is transmitted to the FOCs. Thus, even if the optimization errors are independent the error terms in (40) and (41) are correlated. Since the error vector in the present system is similar to the one in the CET-CD system, the ML procedure will be exactly the same. We can also use the extended Jondrow et al. (1982) formula to estimate observation-specific technical inefficiency. These are not discussed here to avoid repetitions.

Instead of using the system approach if one uses the ODF in (40) alone the estimated parameters will suffer from endogeneity problem since all the regressors are correlated with the error term. The question is whether one can rewrite the ODF so that some of the regressors can be treated as exogenous which does not require instruments. For this first consider the extreme case when the inputs are exogenous. It can be seen from the first $(M - 1)$ FOCs in (41) that solutions of $\hat{y}_m$ will depend on v and therefore will be correlated with the error term in (40). That is, $\hat{y}_m$ cannot be treated as exogenous (which is routinely done in estimating ODF) even when inputs are exogenous. To explore the general case when x_j are not exogenous, we rewrite the FOCs for x_j in (41) in terms of input and output ratios, viz.,

$$\frac{w_j}{w_1}\frac{x_j}{x_1}\exp(\zeta_j - \zeta_1) = \frac{\beta_j + \sum_k \beta_{jk} \ln \hat{x}_k + \sum_m \delta_{mj} \ln \hat{y}_m + (\sum_k \beta_{jk}) \ln x_1}{\beta_1 + \sum_k \beta_{1k} \ln x_k + \sum_m \delta_{m1} \ln \hat{y}_m + (\sum_k \beta_{1k}) \ln x_1} \tag{42}$$

Solution of $\hat{x}_j$ from (42) will depend on input price ratios, $\ln \hat{y}_m, \ln x_1$ and ζ_j. Note that v does not appear in the $J - 1$ FOCs in (42). Thus, $\hat{x}_j$ can be treated as exogenous if one uses instruments for $\ln \hat{y}_m$ and $\ln x_1$. The logic of this argument is that if one estimates the system in (42) using IV for $\ln \hat{y}_m$ and $\ln x_1$, the predicted values of $\ln \hat{x}_j, j = 2, \cdots, J$ can be used as instruments for $\ln \hat{x}_j$. This will reduce number of required instruments to estimate the ODF in (40) by $(J - 1)$. This can be easily seen if we rewrite the ODF in (40) as

$$\begin{aligned}\ln y_1 = \alpha_0 &+ \sum_{j=2}\beta_j \ln \hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln \hat{x}_j \ln \hat{x}_k + \frac{1}{2}\sum_{m=2}\sum_{n=2}\alpha_{mn}\ln \hat{y}_m \ln \hat{y}_n + \sum_{j=2}\sum_{m=2}\delta_{mj}\ln \hat{x}_j \ln \hat{y}_m \\ &+ \sum_{j=m}\alpha_m \ln \hat{y}_m + \ln x_1 \Big\{ \sum_{j=2}[\beta_j + (\sum_k \beta_{jk})\ln \hat{x}_j + \sum_{m=2}\delta_{mj}\ln \hat{y}_m] \Big\} + \{\ln x_1\}^2 \frac{1}{2}\sum_j \sum_k \beta_{jk} + u + v\end{aligned} \tag{43}$$

Note that the presence of production uncertainty makes $\hat{y}_m$ endogenous even if x_j are exogenous. This can be seen from the first $(M-1)$ FOCs in (41) in which v is present. In summary, if the IDF is rewritten in ratio form (as in (43)) it can be estimated first using IV for $\ln \hat{y}_m$ and $\ln x_1$ (ignoring inefficiency). Inefficiency can be estimated in the second step using the residuals of the ODF from the first step..

5.2 The IDF model

Now we consider the IDF model, which after introducing production uncertainty via $\ln A = \alpha_0 + v$ can be written as

$$\begin{aligned}\ln x_1 = \alpha_0 &+ \sum_{j=2}\beta_j \ln \hat{x}_j + \frac{1}{2}\sum_{j=2}\sum_{k=2}\beta_{jk}\ln \hat{x}_j \ln \hat{x}_k + \sum_m \alpha_m \ln y_m \\ &+ \frac{1}{2}\sum_m \sum_n \alpha_{mn}\ln y_m \ln y_n + \sum_m \sum_{j=2}\delta_{mj}\ln \hat{x}_j \ln y_m + u + v\end{aligned} \tag{44}$$

Write $x_1 = x_1^e\, e^v/\mu$ where $x_1^e = E(x_1|u)$ and $\mu = E(e^v)$ so that $E(\pi) = \sum_m p_m y_m - w_1 x_1^e - \sum_{j=2} w_j x_j$. The FOCs with respect to $y_m, m = 1, \cdots, M$ and $x_j, j = 2, \ldots, J$ are

$$\begin{aligned}\frac{\partial \ln x_1}{\partial \ln y_m} &= \frac{p_m}{w_1}\frac{y_m}{x_1}\frac{1}{\mu}\exp(\xi_m + v) = \alpha_m + \sum_n \alpha_{mn}\ln y_n + \sum_j \delta_{mj}\ln \hat{x}_j \\ -\frac{\partial \ln x_1}{\partial \ln \hat{x}_j} &= \frac{w_j}{w_1}\frac{x_j}{x_1}\frac{1}{\mu}\exp(\zeta_j + v) = -[\beta_j + \sum_k \beta_{jk}\ln \hat{x}_k + \sum_m \delta_{mj}\ln y_m]\end{aligned} \tag{45}$$

where ξ_m and ζ_j are optimization error with respect to y_m and x_j, respectively. Logarithms of these $(M-1+J)$ FOCs along with the IDF in (44) define the system in which the endogenous variables are $\ln y_m, \ln x_1$ and $\ln \hat{x}_j$. Note that the uncertainty term v is transmitted to the FOCs. Thus, even if the optimization errors are independent the error terms in (44) and (45) are correlated. The error vector in the present system is identical to the one in the ODF case. So the ML procedure to estimate the parameters and the extended Jondrow et al. (1982) will be exactly the same. Because of this similarity we are not discussing it in details.

Instead of using the system approach if one uses the IDF in (44) alone, the parameter estimates will suffer from endogeneity problem as before. The question is whether one can rewrite the IDF so that some of the regressors can be treated as exogenous which does not require instruments. For this we rewrite the IDF in (44) as

$$\begin{aligned}\ln x_1 = {} & \alpha_0 + \sum_{j=2} \beta_j \ln \hat{x}_j + \frac{1}{2} \sum_{j=2} \sum_{k=2} \beta_{jk} \ln \hat{x}_j \ln \hat{x}_k + \frac{1}{2} \sum_{m=2} \sum_{n=2} \alpha_{mn} \ln \hat{y}_m \ln \hat{y}_n + \sum_{j=2} \sum_{m=2} \delta_{mj} \ln \hat{x}_j \ln \hat{y}_m \\ & + \sum_{m=2} \alpha_m \ln \hat{y}_m + \ln y_1 \sum_m \left\{ \alpha_m + (\sum_{n=2} \alpha_{mn}) \ln \hat{y}_n + \sum_{j=2} \delta_{mj} \ln \hat{x}_j \right\} + \{\ln y_1\}^2 \frac{1}{2} \sum_m \sum_n \alpha_{mn} + u + v\end{aligned} \tag{46}$$

and check whether input ratios are exogenous or not. If we assume that outputs are exogenous the solutions of $\hat{x}_j$ will depend on v (follows from the last $(M-1)$ equations in (45)), which means use of the standard IDF will give inconsistent parameter estimates. To examine whether the input and output ratios in (46) can be treated as exogenous in the general case, we rewrite the FOCs for y_m in (45) in terms of input and output ratios, viz.,

$$\frac{p_m}{p_1} \frac{y_m}{y_1} \exp(\xi_m - \xi_1) = \frac{\alpha_m + \sum_n \alpha_{mn} \ln \hat{y}_n + \sum_j \delta_{mj} \ln \hat{x}_j + (\sum_n \delta_{mn}) \ln y_1}{\alpha_1 + \sum_n \alpha_{1n} \ln \hat{y}_n + \sum_j \delta_{1j} \ln \hat{x}_j + (\sum_n \delta_{1n}) \ln y_1} \tag{47}$$

It can be seen that these FOCs do not depend on v. Thus, $\hat{y}_m$ can be treated as exogenous if one uses instruments for $\ln y_1$ and $\ln \hat{x}_j$. The logic for this argument, as before, is that predicted values of $\ln \hat{y}_m$ from the system in (47) using IV for $\hat{y}_1$ and $\ln x_j, j = 2, \cdots, J$ can be used as instruments for $\ln \hat{y}_m$. This will reduce number of required instruments to estimate the ODF by $(M-1)$.

In summary, because of endogeneity of regressors a single-step SF approach cannot be used. The IDF has to be estimated using system IV or system GMM in the first step. The residuals of it can then be used in the second step to estimate inefficiency using the standard frontier technique.

6 Data

The data source for this study is the Norwegian Farm Accountancy Survey. This is farm-level data collected by the Norwegian Agricultural Economics Research Institute. The data is a large unbalanced panel with 4678 observations on 714 dairy farms observed during 1992 to 2006. It includes farm production and economic data collected annually from about 1000 farms from different regions, farm size classes, and types of farms.

Participation in the survey is voluntary. There is no limit on the number of years a farm may be included in the survey. Approximately 10% of the farms surveyed are replaced every year. The farms are classified according to their main category of farming, defined in terms of the standard gross margins of the farm. Thus the main share of the total standard gross margin for farms categorized as dairy farms comes from dairy production.

Dairy farms are usually involved in other farm production activities such as production of various types of meat, crop, etc. We consider two outputs: milk measured in liters of milk sold (Y_1), and a single measure of all other outputs (Y_2) which include beef and cattle, pigs, sheep, goats and crop products. Since it includes several outputs we measure it in value terms, i.e., revenue from all these outputs. The nominal values are then converted to 2006 values by deflating them using appropriate price indexes. We use the following inputs: land measured in decares (daa = 0.1 hectare) (X_1), own and hired farm labor in hours (X_2), purchased feed (X_3), materials (X_4) (which include cost of fertilizer, pesticides, preservatives, cost related to animal husbandry, etc.). Purchased feed and materials are measured in 2006 values. Both are converted to 2006 values by deflating them using their price indices. The price information is taken from the farm survey when available, and when not, from the agricultural sector of the national accounts.[16]

7 Results

Since panel data is used to estimate the models, we added a time trend variable in all the models proposed before. This time trend variable is introduced to capture technical change (TC). TC in the transformation function can be viewed in two ways: first, a proportional change in all outputs holding inputs unchanged; second, proportional change in all inputs holding outputs unchanged. For technical progress, the first measure should be positive (output augmenting) while the second measure should be negative (input/cost diminution). Thus, in the case of CET-CD transformation function if the coefficient of the time trend is negative, it would indicate output augmenting technical progress (i.e., $TC_O = \partial \ln y_m / \partial t \; \forall \; m > 0$). On the other hand, if the coefficient on the time trend variable in the ODF specification is positive then TC is output augmenting. This is because TC_O in terms of the ODF is simply $\partial \ln y_1 / \partial t$. Thus, if the coefficient is negative it represents technical regress. For the IDF specification it is the opposite. Also, $TC_I = \partial \ln x_j / \partial t \; \forall \; j$ in the transformation function is equivalent to $TC_I = \partial \ln x_1 / \partial t$ in the IDF, and if it is negative then we have technical progress (input or cost diminution). Thus, for technical progress the coefficient of time in the IDF (ODF) formulation of the CET-CD case will be negative (positive). To avoid this confusion researchers

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[16] Details about the data can be found in Kumbhakar et al. (2008).

often define TC in the IDF as minus TC_I so that positive (negative) numbers in both cases mean technical progress (regress). However, interpretation of TC_O and $-TC_I$ are different (one is output augmenting while the other one is cost diminishing). Thus, even if both TC_O and $-TC_I$ have the same value, they do not mean the same thing.

In the TL case we use the same definitions for TC_O and TC_I although they are now functions of data and parameters and are therefore observation-specific.

In addition to computing TC we also calculate returns to scale (RTS) for both the CET-CD and TL specifications. In terms of the transformation function RTS is defined as (Panzar and Willig (1977)) $RTS = -\sum_j \partial \ln f(\cdot)/\partial \ln x_j \div \sum_m \partial \ln f(\cdot)/\partial \ln y_m$. Thus RTS in the CET-CD case (similar to TC) is a constant (same for all observations) while it is observation-specific in the TL case.

Using the above definitions we compute RTS and TC from both the CET-CD and TL models. The input distance function specification of the CET-CD model failed to satisfy the requirement that the parameter c is greater than unity (which means that it failed to satisfy the second order condition of profit maximization). Because of this we decided not to report any results from the IDF model. Such violations did not occur in the ODF and system models. Note that RTS and TC in the CET-CD models are constant (same for all observations). This is true irrespective of whether one considers the single or system models. RTS in the CET-CD output distance function is 0.98 and technical change (output augmentation) is 0.27% per annum. RTS and TC from the CET-CD system are 0.81 and 1.45%, respectively. Thus moving from the ODF to the system approach reduced average RTS but increased TC.

Results from the translog models are more interesting because the estimates of RTS and TC are observation-specific. For the translog case, we report results from the ODF, IDF and system models. The mean RTS for these models are 0.74, 0.83 and 0.81, respectively. There are only a few observations where RTS exceeded unity. Histograms of RTS for these models are reported in Figures 1-3. It can be seen from these histograms that there are substantial variations in RTS in these models. The pattern is quite similar across these three models.

So far as technical change is concerned, we find negative values of TC (technical regress) for slightly less than 50% of the observations. This is especially the case in both the IDF and ODF models. However, in the system model the percentage of farms that experienced technical regress is somewhat lower. The mean TC in the IDF, ODF and system models are 0.17%, -0.22% and 0.52%, respectively. Although the mean values are quite low, the distributions of TC show that there are some farms that experienced technical progress of more than 2% per annum. Details of TC for each of these models can be seen from Figures 4-6. Substantial variations in TC is observed among farms both within and between models. Note that TC in the IDF model has a different interpretation which is different from those in the ODF and system models.

Finally, we examine technical efficiency in each of these models. Mean TE in the CET-CD output distance function model is 0.87 with the minimum and maximum values 0.58 and 0.98. The corresponding figures for the system model are: mean 0.80, minimum 0.42 and maximum 0.93.[17] Frequency distribution of TE for these models are reported in Figures 7 and 8. Distributions of TE for these two models are quite similar, except for the fact that efficiency values are slightly lower in the system model. While examining TE in the translog models we find that mean technical efficiencies are somewhat higher compared to those in the CET-CD models (except for the ODF model). The mean TE for the translog ODF, IDF and system models are: 0.86, 0.93 and 0.94, respectively. Frequency distributions of TE from these models are reported in Figures 9-11. It can be seen that these distributions are quite similar, except for some differences in the level. In general, we find that distributions of TE are quite robust to alternative model specifications.

8 Conclusion

Distance functions are widely used in production economics, especially in the efficiency literature. Conventional wisdom dictates one to use the IDF (ODF) when outputs (inputs) are exogenous. This is because when outputs are exogenous, the first order conditions for profit maximization is the same as those of cost minimization and the solution of input ratios from these first-order conditions are exogenous (not affected by inefficiency). Thus, all the regressors in an IDF model are exogenous (uncorrelated with inefficiency). Similarly, if inputs are exogenously given, output ratios for a profit maximizing (revenue maximizing) firm will be exogenous. This makes all the regressors in an ODF model exogenous. However, in many applications it is not clear whether outputs (inputs) are exogenous or not. That is, if inputs and outputs are choice (decision) variables to the farm whether input and output ratios will be correlated with the error term in the estimating equation depends on what is in the error term and whether it is known to the farm. So the question is: Can one treat input ratios (output ratios) as exogenous in an IDF (ODF) if both inputs and outputs are choice variables? If so under what conditions? Alternatively, should one be concerned about endogeneity of only outputs (inputs) in the IDF (ODF), or both input (output) ratios as well as outputs (inputs)? We discussed these issues in details in this paper in terms of different specifications (production function, IDF, ODF and input requirement function).

If input and output prices are available one can use them as instruments and apply system IV/GMM to get consistent estimates from different specifications. However, since the regressors (and their endogeneity)

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[17] Efficiency in the system models are estimated using the two-step procedure.

depend on the specification one uses, requirements of instruments also vary. Even if there are enough instruments available the single-step stochastic frontier approach cannot be used in the single equation IDF and ODF models. One has to use the two-step procedure to estimate efficiency. The other alternative is to use a system approach which takes into account endogeneity of inputs and outputs explicitly. The advantage of using the system approach is that it is invariant to whether an ODF or an IDF or a production function is used. System approach can also be used when outputs (inputs) are exogenous. Finally, observation-specific technical efficiency can be estimated directly using the parameter estimates. We discussed these issues in details for both the restrictive (constant elasticity of transformation output function and the Cobb-Douglas input function) and flexible (non-separable translog) parametric transformation functions.

As an empirical illustration we used Norwegian dairy farming data and estimated IDF, ODF, and the system models. First, we used the constant elasticity of transformation output function and the Cobb-Douglas input function. The models are then extended to accommodate translog specifications without making the input function separable from the output function. In spite of some differences in the estimates of returns to scale and technical change across different models, estimated technical efficiency are found to be quite high and robust.

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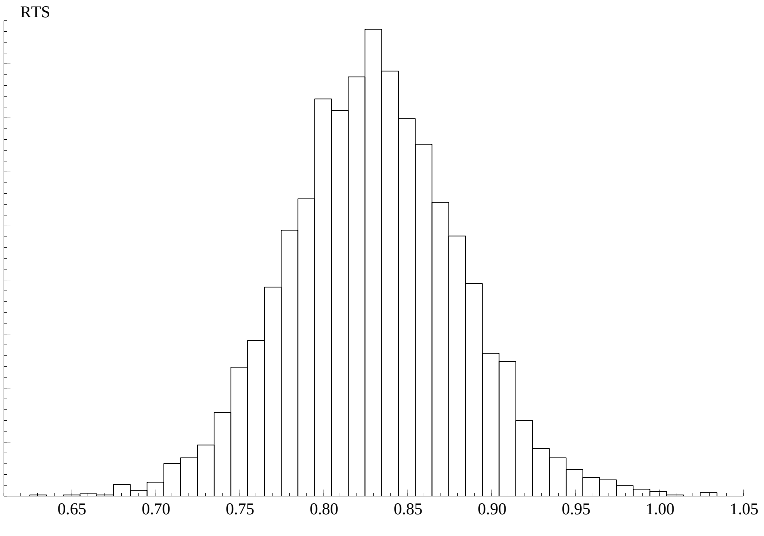

Figure 1: RTS in the translog IDF model

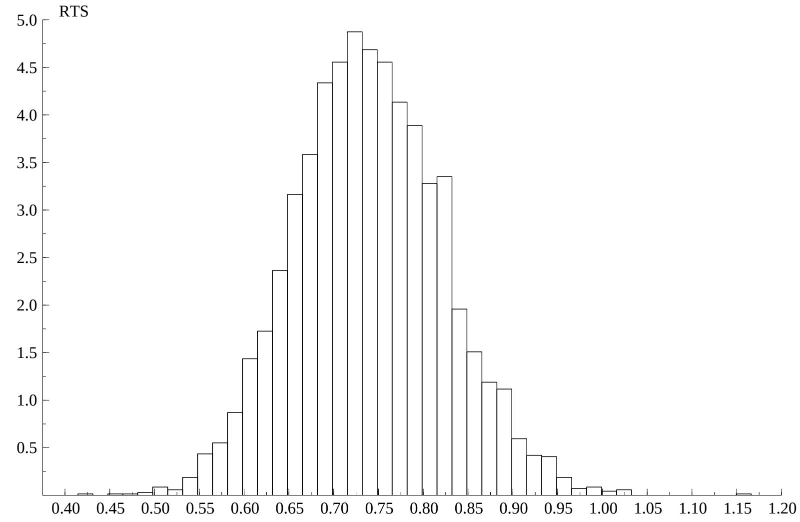

Figure 2: RTS in the translog ODF model

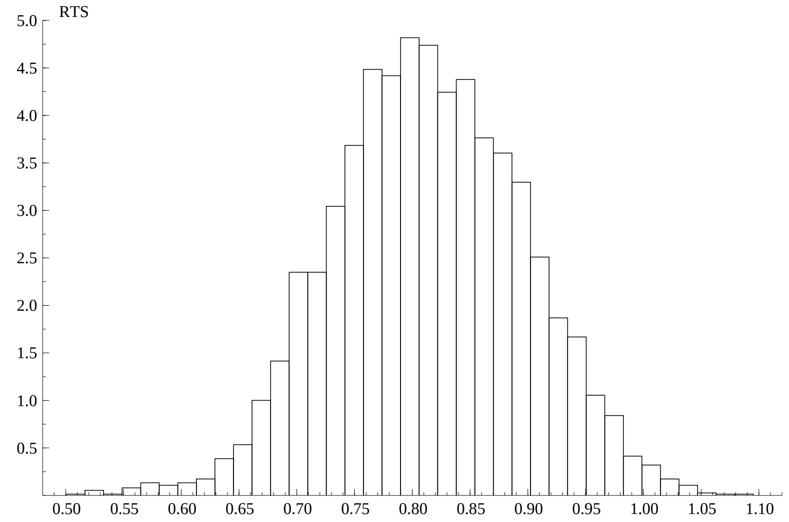

Figure 3: RTS in the translog IDF system model

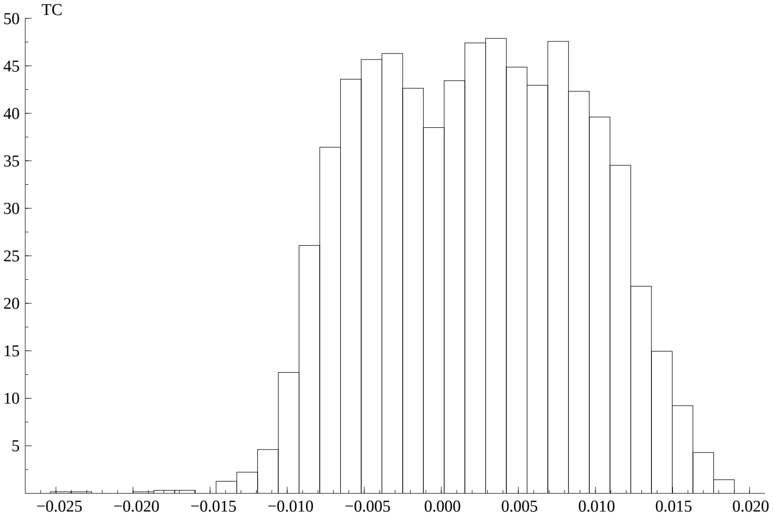

Figure 4: TC in the translog IDF model

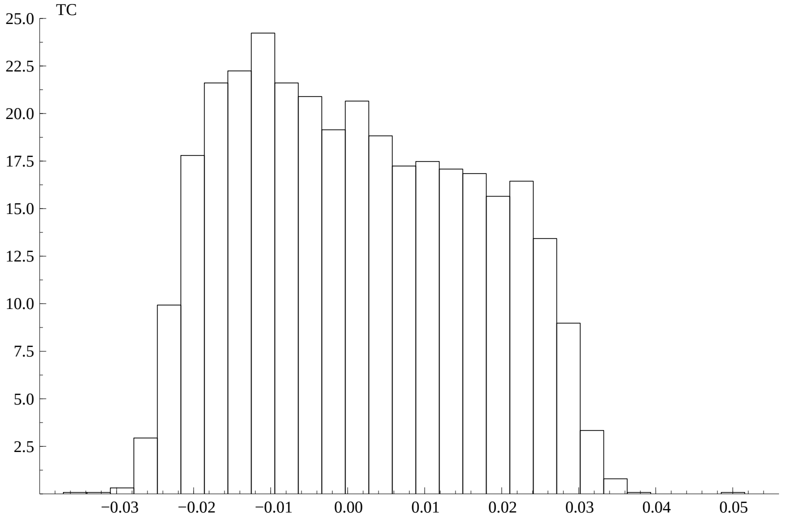

Figure 5: TC in the translog ODF model

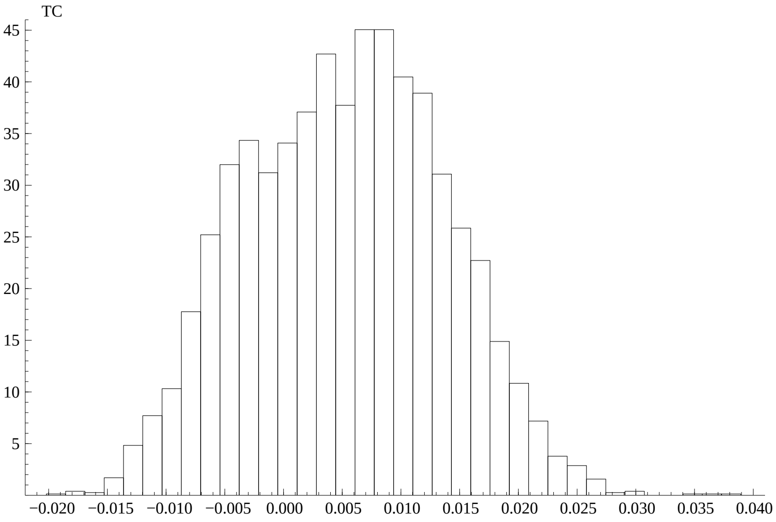

Figure 6: TC in the translog system model

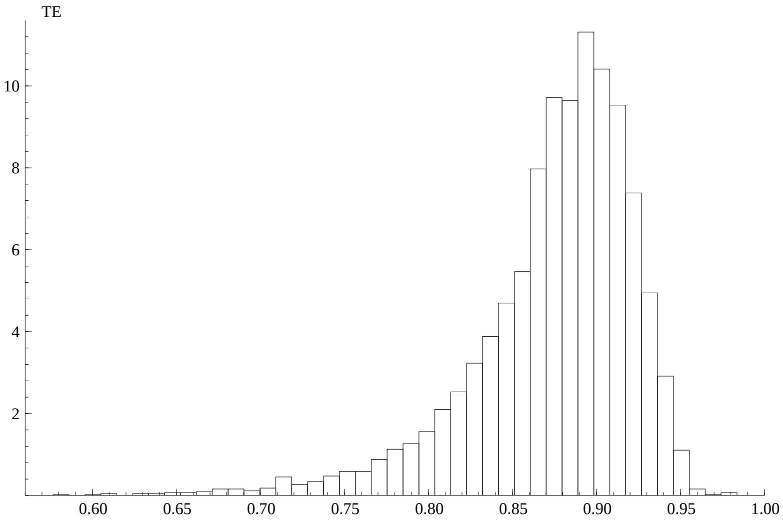

Figure 7: Technical efficiency in the CET-CD production model

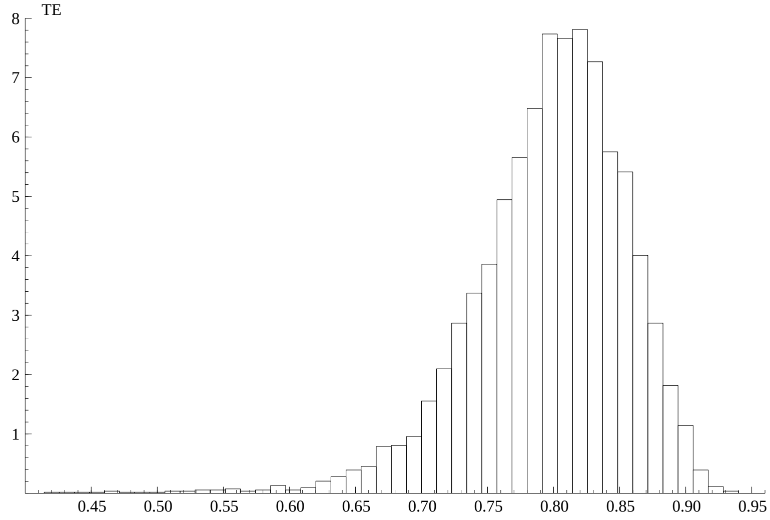

Figure 8: Technical efficiency in the CET-CD system model

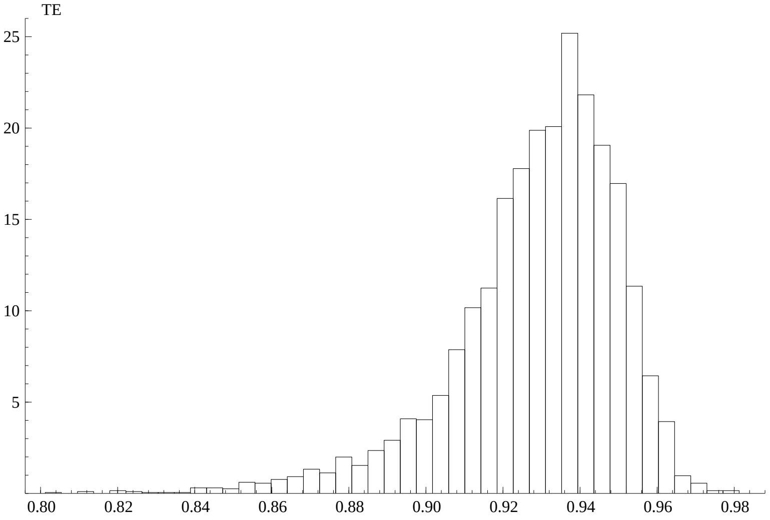

Figure 9: Technical efficiency in the translog IDF model

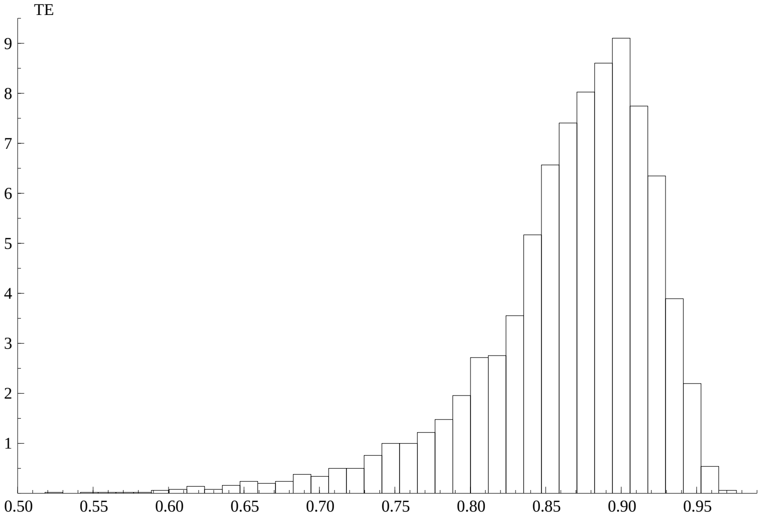

Figure 10: Technical efficiency in the translog ODF model

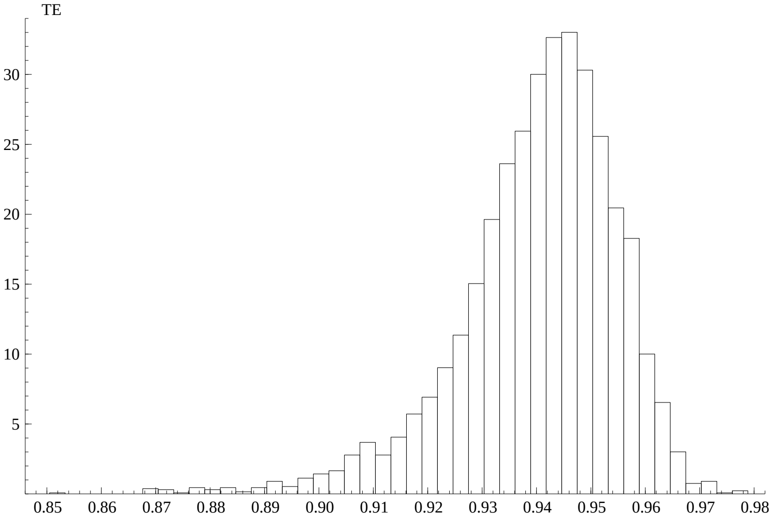

Figure 11: Technical efficiency in the translog system model